ALTERNATIVE CONCEPTS AND ALGORITHMS OF MIN – MAX ROBUSTNESS FOR UNCERTAIN MULTICRITERIA OPTIMIZATION PROBLEMS

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ABSTRACT

The aim of this thesis is to introduce new solution concepts regarding the uncertainty of multicriteria optimization problems. In the first approach, we concern with the practical implementation of problems in which the solution should satisfy priority levels in the objective function and the worst performance vector of the solution obtained by the proposed concept is close to a reference point of the considered problem, within an acceptable tolerance threshold. To this aim, the concept of lexicographic tolerable robust solution is suggested. Additionally, important properties of the solution sets of this introduced concept as well as an algorithm for finding such solutions are presented and discussed. The second approach, we focus on the implementation of problems in which the solution should provide the optimal choice concerning the possible maximum criteria on the worst case scenario, at the same time a solution obtained by the proposed concept should provide a good balance in the quality between the uncertain situation and the undisturbed situation of the considered problem. To this aim, the lightly robust max-ordering solution is introduced. In order to support the decision maker in understanding the trade-off between robustness and quality in undisturbed situation of the lightly robust max-ordering solution, two measures called 'gain in robustness' and

'price to be paid for robustness' are presented. Moreover, an algorithm for finding the proposed solution concept is presented and discussed.

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CHAPTER I

INTRODUCTION

Problem solving and decision making belong together in our lives, without making a decision that problem cannot be solved. There are many reasons that the method in mathematical optimization techniques often do not apply to real-world problems. This is because problem solving in many cases must consider the problem bearing difficulties that are included in the structure of the problem itself. The two main obstacle reasons of these solving problem bearing difficulties are, that most real-world problems are of multicriteria in nature, and the input data is usually not known beforehand or available information is subject to change. The lack of available information limits our ability to make choices that may have an impact of our daily lives. Some motivation situations can be shown in the following examples: are considering buying a car, the purposes of this decision are to obtain a car with highest comfort and with the lowest energy consumption for the cheapest price; nevertheless, the car with highest comfort and the cheapest cost value often conflicts with consumers objectives. Moreover, the decision on whether to buy or not has to be made basically at the beginning time you considering buying, for an individual cannot predict future product designs or price of gas changes. Therefore, this purchasing activity, the sometime mundane task of choosing a car, is a decision optimizing multiple objectives under uncertainty.

Another example is when a farmer, who is in the process of deciding which crops to grow in the coming years after a short rotational period, needs to manage his crop planting selections. If a farmer intends to cultivate a specific crop in order to obtain the maximum harvest but is not sure of which crop should be grown, or what the weather conditions will be in the upcoming years, farmers need to consider which crop they should grow. Their decision must incorporate which crop will minimize the cost of cultivation and minimizing harvesting difficulties, while maximizing the profits. Therefore, deciding which crops to cultivate requires optimizing multiple objectives under uncertainty.

Another example concerns of water management issues. Imagine you are a engineer or maybe Politian etc., that has to decide which plan to choose regarding the construction of a controversial dam. The major issues that could arise can be seen in maximizing energy consumption, minimizing construction costs, and minimizing negative effects on population resettlement. Addtionally, individual are not able to predict the weather or the price of purchasing energy consumption that is regulated by future government policies. This means individuals in these circumstances face the problem of multi-goal optimization under uncertainties.

As we pointed out above, optimizing multicriteria in the event of uncertainty is applicable in different areas of the real world. Individual mathematical communities have developed multicriteria optimization techniques and practices for dealing with uncertainties concerning problem formulation. The idea of optimizing objectives in the event of uncertainty can be effectively applied to a diverse number of issues in different areas of the real world.

As mentioned above, many real-world optimization problems are often multifaceted in nature and involve uncertainties in the input of data regarding a problem. As a consequence, researchers are attempting to develop mathematical modelling and methods to solve the problem. Motivated by the significance of these problems, in this thesis, we are going to introduce new concepts of robustness for multicriteria optimization problems under uncertainty situation.

The structure of the thesis is as follows

Chapter I. This chapter is an introduction to the research problems.

Chapter II. In this chapter, the basic concepts, notations, and solution techniques of deterministic multicriteria optimization and uncertain multicriteria optimization will be repeated and introduced. Also, an introduction to the various approaches that can be found in the literature to handle uncertain data in both single objective and multicriteria problems are recalled and presented.

Chapter III. In this chapter, we present a new concept of robustness for uncertain multicriteria optimization problems which is a generalisation of the solution concept in multicriteria optimization problems, which we are known as a lexicographic solution. In section 3.2 and section 3.3, important properties and an algorithm for finding the proposed solution are presented and discussed, respectively. In section 3.5, the implementation of our proposed solution is presented to a water resource management problem.

Chapter IV. In this chapter, we present a new concept of robustness for uncertain multicriteria optimization problems with the generalisation of a solution concept based on multicriteria optimization problems, which we are known as a max-ordering solution. In section 4.1, we present a fundamental on the set of our proposed solution concept. In section 4.2, we introduce the price of robustness for the proposed solution and present the method for finding the proposed solution to assist the decision maker in making decision process. In section 4.3, we provide a demonstration for the proposed solution concept on the ambulance location problem with uncertainty situation.

Chapter V. We give the concluding research.

CHAPTER II

PRELIMINARIES

In this chapter, we present notations and basic concepts which will be used throughout this work. We also give a literature review of uncertain multicriteria optimization problems. Moreover, various approaches in the literature to handle uncertain data in the problems are recall and discussed.

We now introduce the main notations and basic concepts. From now on, the set of all natural numbers and the set of all real numbers will be denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. For each $p \in \mathbb{N}$, notations $\mathbb{R}^p$ and I_p are used to stand for a vector space with p dimensions and the index set $\{1, 2, \ldots, p\} \subseteq \mathbb{N}$, respectively. For vectors $x, y \in \mathbb{R}^p$ with $x = (x_1, x_2, \ldots, x_p)$ and $y = (y_1, y_2, \ldots, y_p)$, we define the relations $\precapprox, \preccurlyeq, \prec$, and $\leq_{lex}$ as follows:

$$x \precapprox y \Leftrightarrow x_i \leqslant y_i \text{ for all } i \in I_p,$$

$$x \preccurlyeq y \Leftrightarrow x_i \leqslant y_i \text{ for all } i \in I_p \text{ and } x \neq y,$$

$$x \prec y \Leftrightarrow x_i < y_i \text{ for all } i \in I_p,$$

$$x \leq_{lex} y \Leftrightarrow \text{if } x_m < y_m \text{ where } m := \min\{k | x_k \neq y_k\} \text{ and } x_i = y_i \text{ for all } i \in I_{m-1}.$$

Remark 2.0.1. The notations $\precapprox, \preccurlyeq, \prec$, and $\leq_{lex}$ are used to stand for the relations on $\mathbb{R}^p$ and $\leqslant, <$ are used to stand for the relations on $\mathbb{R}$, respectively.

By using the inverse inequality analogy, including $\gtrapprox, \succcurlyeq, \succ$, and $\geq_{lex}$, we define the definition of cones with respect to each relation as follows. For each $p \in \mathbb{N}$, the orthants of $\mathbb{R}^p$ with respect to $\gtrapprox, \succcurlyeq, \succ$, and $\geq_{lex}$ are defined by

$$\mathbb{R}^p_{\gtrapprox} := \{x \in \mathbb{R}^p | x \gtrapprox 0\},$$

$$\mathbb{R}^p_{\succcurlyeq} := \{x \in \mathbb{R}^p | x \succcurlyeq 0\},$$

$$\mathbb{R}^p_{\succ} := \{x \in \mathbb{R}^p | x \succ 0\},$$

$$\mathbb{R}^p_{\geq_{lex}} := \{x \in \mathbb{R}^p | x \geq_{lex} 0\}.$$

2.1 Solution concepts for deterministic multicriteria optimization

In this section, we recall deterministic multicriteria optimization problems and also various approaches of solution concepts for the problem.

Definition 2.1.1. A *multicriteria optimization problem* $\mathcal{MP}$ is a problem dealing with minimizing vector-valued objective function $f : \mathbb{R}^n \to \mathbb{R}^p$ over some constraint $X \subseteq \mathbb{R}^n$. That is,

$$(\mathcal{MP}) \qquad \min f(x) \qquad \text{subject to } x \in X. \tag{2.1.1}$$

Remark 2.1.2. Note that if $p = 1$, then the problem $\mathcal{MP}$ is called a single objective optimization problem.

When $p \geqslant 2$, a solution that minimizes all objective functions simultaneously does usually not exists. Consequently, it is necessary to define a minimum sense with regard to the problem-solving approach. According to the above mentioned relations, various approaches of solution concepts for the problem $\mathcal{MP}$ are introduced. Here, we present some well-known solution concepts which was introduced by Vilfredo Pareto in 1896 [1].

Definition 2.1.3. [**Efficiency concept**] Given a multicriteria optimization problem $\mathcal{MP}$ and feasible set X. A feasible solution $\hat{x} \in X$ is called an *efficient solution* or *Pareto solution* if there is no $x' \in X$ such that

$$f(x') \ \preccurlyeq \ f(\hat{x}). \tag{2.1.2}$$

By replacing the relation $\preccurlyeq$ with $\lesssim$ or $\prec$ in the inequality (2.1.2), a feasible solution $\hat{x}$ is called a *strictly efficient solution* and *weakly efficient solution*, respectively. The set of all efficient solutions, strictly efficient solutions, and weakly efficient solutions of the problem $\mathcal{MP}$ will be denoted by X_E, X_{sE} , and X_{wE}, respectively.

Remark 2.1.4. We note that the solution corresponding to the essential characteristics of the concept of efficiency will provide a trade-off among all objectives.

Definition 2.1.5. [**Lexicographically optimality concept**] Given a multicriteria optimization problem $\mathcal{MP}$ and feasible set X. A feasible solution $\hat{x} \in X$ is called *lexicographically optimal* or a *lexicographic solution* if there is no $x' \in X$ such that

$$f(x') \ \leq_{lex} \ f(\hat{x}). \tag{2.1.3}$$

The set of all lexicographic solutions of the problem $\mathcal{MP}$ is denoted by X_{lex}.

Remark 2.1.6. The lexicographic optimality in Definition 2.1.5 provides a solution in which satisfy the priority levels in the objective function in the sense that the first criterion f_1 has the highest priority, and only case of multiple optimal solutions the criterion f_2 and the further criteria are considered.

Definition 2.1.7. [**Max-ordering optimality concept**] Given a multicriteria optimization problem $\mathcal{MP}$ and feasible set X. A feasible solution $\hat{x} \in X$ is called a *max-odering solution* if there is no $x' \in X$ such that

$$\max_{k\in I_p} f_k(x') \ < \ \max_{k\in I_p} f_k(\hat{x}). \tag{2.1.4}$$

The set of all max-odering solutions of the problem $\mathcal{MP}$ is denoted by X_{MO}

Now, we recall the technique for order preference by similarity to ideal solution (TOPSIS) and the computational procedure for the TOPSIS method which was proposed by Hwang and Yoon [2].

Definition 2.1.8. [TOPSIS] Given a multicriteria optimization problem $\mathcal{MP}$ and finite feasible set $X = \{x_1, x_2, \ldots, x_n\}$. Let $w_1, w_2, \ldots, w_p$ be the positive weights of the component objective functions $f_1, f_2, \ldots, f_p$, respectively, such that $\sum_{j=1}^{p} w_j = 1$. The TOPSIS method for finding the solution of the considered problem is following.

Initialization. Input a multicriteria optimization problem $\mathcal{MP}$.

Step 1. For each $i \in I_n$, compute the weighted normalized value r_{ij} of component function f_j of each feasible solution x_i by the following formulas,

$$r_{ij} = w_j \frac{f_j(x_i)}{\sqrt{\sum_{i=1}^{n} f_j(x_i)}}, \text{ for all } j \in I_p.$$

We write vector of weight normalized value of each feasible solution x_i by $(r_{i1}, r_{i2}, \ldots, r_{ip})$, for all $i \in I_n$.

Step 2. Determine the positive ideal solution A^+ and the negative ideal solution A^- by using the formulas,

$$\begin{aligned} A^+ &:= (a_1^+, a_2^+, \ldots, a_p^+) \\ &= \left(\max_{i \in I_n} r_{i1}, \max_{i \in I_n} r_{i2}, \ldots, \max_{i \in I_n} r_{ip} \right), \end{aligned} \tag{2.1.5}$$

and

$$\begin{aligned} A^- &:= (a_1^-, a_2^-, \ldots, a_p^-) \\ &= \left(\min_{i \in I_n} r_{i1}, \min_{i \in I_n} r_{i2}, \ldots, \min_{i \in I_n} r_{ip} \right). \end{aligned} \tag{2.1.6}$$

Step 3. Compute the distances from the normalized vector $(r_{i1}, r_{i2}, \ldots, r_{ip})$ of each feasible solution x_i to the positive ideal solution A^+ and A^- by using the following formulas,

$$d_i^+ = \sqrt{\sum_{j=1}^{p}(r_{ij} - a_j^+)^2} \text{ for all } i \in I_n,$$

and

$$d_i^- = \sqrt{\sum_{j=1}^{p}(r_{ij} - a_j^-)^2} \text{ for all } i \in I_n.$$

Step 4. Compute the relative distance D_i^* of each feasible solution x_i respecting the positive ideal solution A^* and the negative ideal solution A^- by the formula,

$$D_i^* = \frac{d_i^-}{d_i^+ + d_i^-}, \text{ for all } i \in I_n.$$

Step 5. Find the *solution* x^B which is determined by

$$x^B := \underset{i \in I_n}{argmax} D_i^*,$$

Motivated by the lexicographic solution concept in Definition 2.1.5 and the max-ordering solution concept in Definition 2.1.7, new solution concepts for uncertain multicriteria optimization problems will be introduced in the Chapter III and the Chapter IV, respectively. In addition, the relations between existing solution concepts and proposed solution concepts will be analyzed and discussed.

2.2 Solution concepts for uncertain multicriteria optimization

In this section, we recall uncertain multicriteria optimization problems and robustness concepts which are motivated and related to our proposed solution concepts.

Definition 2.2.1. An *uncertain multicriteria optimization problem* $\mathcal{MP}(\mathcal{U})$ is given as a family of $\{\mathcal{MP}(s)|s \in \mathcal{U}\}$ of deterministic multicriteria optimization problems

$$(\mathcal{MP}(s)) \qquad \min f(x,s) \qquad (2.2.1)$$

$$\text{subject to } x \in X$$

with the objective function $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^p$, feasible set $X \subseteq \mathbb{R}^n$, and uncertainty set $\mathcal{U}$. An element $s \in \mathcal{U}$ indicates a particular value for the uncertain parameters belonging in an uncertainty set $\mathcal{U}$.

Remark 2.2.2. When $p = 1$, an uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$ is an uncertain single objective optimization problem.

To decide what is a good solution for the problem $\mathcal{MP}(\mathcal{U})$ is not easy. Some of the first research done in the area of uncertain multicriteria optimization was the solution concept introduced by Deb and Gupta [3] in 2006. They replaced the objective vector in a given uncertain multicriteria optimization problem with the mean effective functions computed by averaging a representative set of neighboring solutions, thereby removing the uncertainly and converting the problem to just a deterministic multicriteria optimization problem. Then, an efficient solution for that deterministic multicriteria optimization problem is considered as a robust solution for the full original uncertain multicriteria optimization problem.

In the following subsection, we present some important concepts that will serve to motivate our ideas.

2.2.1 Minmax robustness

Instead of using the concept of mean effective functions which were considered in [3], Kuroiwa and Lee [4] reformulated uncertain multicriteria optimization problems $\mathcal{MP}(\mathcal{U})$ by replacing the objective vector in the original problem with a vector consisting of the worst case scenario of each respective component. Therefore, yielding a deterministic multicriteria optimization problem with vector-valued objective function:

$$f^{wc}(x) = \begin{pmatrix} \sup\limits_{s\in\mathcal{U}} f_1(x,s) \\ \sup\limits_{s\in\mathcal{U}} f_2(x,s) \\ \vdots \\ \sup\limits_{s\in\mathcal{U}} f_p(x,s) \end{pmatrix}$$

Then, an efficient solution for that deterministic multicriteria optimization problem with respect to the objective function f^{wc} is considered as a robust solution. The formal definition of the robustness concept in [4] is following.

Definition 2.2.3. [4] [**Point-based minmax robust efficiency**] Given an uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$. A feasible solution $\hat{x} \in X$ is called *point-based minmax robust efficient* if there is no $x' \in X\setminus\{\hat{x}\}$ such that

$$f^{wc}(x') \preccurlyeq f^{wc}(\hat{x}).$$

Notice that by replacing the relation $\preccurlyeq$ in Definition 2.2.3 with $\precsim$ or $\prec$, a feasible solution $\hat{x}$ is called *point-based minmax robust strictly efficient* or *point-based minmax robust weakly efficient*, respectively.

Remark 2.2.4. When $p = 1$, the solution approach proposed by Kuroiwa and Lee in Definition 2.2.3 is closely connected to the classical minmax robustness concept for uncertain single objective optimization problems, which was firstly introduced by Soyster [6] and subsequently extensively studied by Ben-Tal and Nemirovski [7]. Remind that a feasible solution $\hat{x}$ is a minmax robust solution for uncertain single objective optimization problems if it is an optimal solution for the deterministic single objective optimization problem

$$\begin{aligned} &\min \sup_{s\in\mathcal{U}} f(x,s) \\ &\text{subject to } x \in X \end{aligned} \tag{2.2.2}$$

with the objective function $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, feasible set $X \subseteq \mathbb{R}^n$, and uncertainty set $\mathcal{U}$.

In 2014, another interpretation of Ben-Tal and Nemirovski's robustness concept was provided by Ehrgott et al. [8]. In this solution concept, for each feasible solution they looked at the set of objective vectors under all scenarios and compared those sets to each other, by using the concept of set relations to define minmax robustness for uncertain multicriteria optimization problems. Now, we recall the formal definition of set-based robust efficiency [8]. To do this, we will denote the set of all possible objective vectors under all scenarios of each feasible solution $x \in X$ with the following notation:

$$f_{\mathcal{U}}(x) := \{f(x,s) : s \in \mathcal{U}\} \subseteq \mathbb{R}^p.$$

Here, the set-based minmax robustness concept in [8] is recalled.

Definition 2.2.5. [8][**Set-based minmax robust efficiency**] Given an uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$. A feasible solution $\hat{x} \in X$ is called *set-based minmax robust efficient* if there is no $x' \in X \backslash \{\hat{x}\}$ such that

$$f_{\mathcal{U}}(x') \subseteq f_{\mathcal{U}}(\hat{x}) - \mathbb{R}^p_{\succcurlyeq}.$$

Notice that by replacing the cone $\mathbb{R}^p_{\succcurlyeq}$ with $\mathbb{R}^p_{\succapprox}$ or $\mathbb{R}^p_{\succ}$ in Definition 2.2.5, the feasible solution $\hat{x}$ is called *set-based minmax robust strictly efficient* or *set-based minmax robust weakly efficient*, respectively.

Remark 2.2.6. Notice that the point-based minmax robust efficiency concept and the set-based minmax robust efficiency concept are identical in the case of considering the objective-wise uncertain multicriteria optimization problem. Here, let us recall that a problem $\mathcal{MP}(\mathcal{U})$ is the *objective-wise uncertain multicriteria optimization problem* if the uncertainties of the objective functions $f_1, f_2, \ldots, f_p$ are independent of each other, namely if $\mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_p$ and for each $(s_1, s_2, \ldots, s_p) := s \in \mathcal{U}$,

$$f(x,s) = \begin{pmatrix} f_1(x,s_1) \\ f_2(x,s_2) \\ \vdots \\ f_p(x,s_p) \end{pmatrix},$$

where $s_k \in \mathcal{U}_k$, for each $k \in I_p$. For more details one may see [8].

A similar approach was introduced by Bokrantz and Fredriksson [9], who used set relations following Ehrgott's work, but replaced the set of objective vectors of a feasible solution under all possible scenarios $f_{\mathcal{U}}(x)$ by its convex hull. Other solution concepts for uncertain multicriteria optimization problems was proposed by Giovanni et al. [10] in 2015 who used set relations on comparing the set $f_{\mathcal{U}}(x)$. We notice that the above four concepts are concerned with minmax robustness, since they hedge against the worst case scenario. For more details on a survey and analysis of different concepts of robustness for uncertain multicriteria optimization, ones may see in [11].

2.2.2 Lightly robustness

As we have seen that the resulting solution of minmax robustness concepts derived by relying on data of the worst case scenario, the decision makers may not be willing to make decisions based on the worst possible one. Moreover, if one wants to hedge against all scenarios from the uncertainty set may consequently come with the high cost. From this point of view, many researchers try to look at the alternative robustness concepts which can be reduced this conservatism of minmax robustness concept. One of all interesting solution concepts that we would refer to is the concept of light robustness. The idea underlying this solution concept is to find solutions that still working concerned robustness, at the same time not too bad in an undisturbed situation from uncertainties concurrently. For a concept of light robustness, a nominal scenario is defined. A scenario is called *nominal* if it is the most typical situation or the most important one among all scenarios in uncertainty set. The original idea of light robustness concept was proposed by Fischetti and Monaci [12] for uncertain single objective optimization problems. In this concept, a feasible solution is said to be a lightly robust solution if its objective value does not differ from the optimal objective value in the nominal scenario more than an acceptable threshold and minimize the objective function for the worst case scenario over all feasible solutions. Now, we recall the formal definition which was introduced by Fischetti and Monaci [12].

Definition 2.2.7. [12][**Light robust optimality for single objective optimization problems**] Given an uncertain single objective optimization problem with nominal scenario $\hat{s}$ and $\varepsilon \geqslant 0$. Assume that x' is an optimal solution to the optimization problem of the nominal scenario $\hat{s}$. Then, a solution $\hat{x} \in X$ is called a *lightly robust optimal solution with respect to* ε to an uncertain single objective optimization problem , if it is an optimal solution to

$$\begin{aligned} &\min \sup_{s \in \mathcal{U}} f(x, s) \\ &\text{subject to } f(x, \hat{s}) \leqslant f(x', \hat{s}) + \varepsilon \\ &\qquad\qquad x \in X, \end{aligned} \tag{2.2.3}$$

with the objective function $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, feasible set $X \subseteq \mathbb{R}^n$, and uncertainty set $\mathcal{U}$.

This solution concept was extensively studied by Kuhn et al. [13] from uncertain single objective optimization problems to uncertain bi-objective optimization problems and then more general setting on uncertain multicriteria optimization problems by Schöbel and Ide [11]. This generalization of lightly robust concept was presented by combining the ideas underlying of the set-based minmax robust efficiency in [8] and lightly robustness concept in [12] together. By replacing the idea of set-based minmax robust efficiency in [8] with the point-based minmax robust efficiency in [4], another interpretation of lightly robust efficiency was presented by Schöbel and Zhou-Kangas [14]. We now recall this solution concept.

Given an uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$ together with a nominal scenario $\hat{s} \in \mathcal{U}$, let $X_E(\hat{s})$ be the set of all efficient solutions to a deterministic multicriteria optimization problem $\mathcal{MP}(\hat{s})$. For each efficient solution $x' \in X_E(\hat{s})$ and some given $0 \preccurlyeq \varepsilon \in \mathbb{R}^p$ with $\varepsilon := (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p)$, a subproblem $\mathcal{LR}(x', \varepsilon, \mathcal{U})$ of the uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$ is defined as a family of deterministic multicriteria optimization problems, that is $\mathcal{LR}(x', \varepsilon, \mathcal{U}) := \{\mathcal{LR}(x', \varepsilon, s) | s \in \mathcal{U}\}$,

$$(\mathcal{LR}(x', \varepsilon, s)) \qquad \min f(x, s) \tag{2.2.4}$$
$$\text{subject to } f_i(x, \hat{s}) \leqslant f_i(x', \hat{s}) + \varepsilon_i, \text{ for all } i \in I_p,$$
$$x \in X.$$

Definition 2.2.8. [14][**Light robust efficiency**] Given an uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$ with nominal scenario $\hat{s} \in \mathcal{U}$ and some given $\varepsilon \in \mathbb{R}^p_{\geqslant}$. A feasible solution $\hat{x} \in X$ is called *lightly robust efficient for* $\mathcal{MP}(\mathcal{U})$ *with respect to* ε if it is point-based minmax robust efficient of the problem $\mathcal{LR}(x', \varepsilon, \mathcal{U})$.

In [14], the theoretical point of view on the relationships between the point-based minmax robust efficient solution concept in [4] and the lightly robust efficient solution concept in [14] are analyzed and compared under the nominal scenario case and the worst case. In addition, the authors also analyzed the trade-off between nominal quality and robustness of a single solution by introducing a measure which is called a *price of robustness.* In this measure, two strategies were presented to support a decision maker in finding the most desirable solution for multicriteria problems in uncertain situations, namely, the 'gain in robustness' and the 'price to be paid for robustness'. By applying these two strategies of the price of robustness in the decision making process, a visualization of the implementation of the proposed solution concept was illustrated to the problem of investment portfolio optimization in Zhou-Kangas and Miettinen [15].

Motivated by the ideas of lexicographic solution in Definition 2.1.5 and the minmax robustness concept in Section 2.2.1, the first approach of robust solution concept for uncertain multicriteria optimization problems will be introduced in Chapter III. Later on, by adoping the ideas of max-odering optimality in Definition 2.1.7 and the light robustness in Section 2.2.2, the second approach of robust solution concept will be introduced in Chapter IV.

2.3 Auxiliary concepts and results

To present the results on relationships between our proposed solution concept and the set-based minmax robust efficiency in Chapter III, we recall the following supplementary result.

Lemma 2.3.1. [8] Given an uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$. Then, the following statements hold.

(a) For all $x', \bar{x} \in X$,

$$f_{\mathcal{U}}(x') \subseteq f_{\mathcal{U}}(\bar{x}) - \mathbb{R}^p_{[\overset{\succsim}{\approx}/\succneq/\succ]} \Longleftrightarrow f_{\mathcal{U}}(x') - \mathbb{R}^p_{\overset{\succsim}{\approx}} \subseteq f_{\mathcal{U}}(\bar{x}) - \mathbb{R}^p_{[\overset{\succsim}{\approx}/\succneq/\succ]}.$$

(b) For all $x', \bar{x} \in X$,

$$f_{\mathcal{U}}(x') - \mathbb{R}^p_{\overset{\succsim}{\approx}} \subseteq f_{\mathcal{U}}(\bar{x}) - \mathbb{R}^p_{[\overset{\succsim}{\approx}/\succneq/\succ]} \Longleftrightarrow \forall s \in \mathcal{U}, \exists s' \in \mathcal{U} : f(x', s)[\overset{\precsim}{\approx} \,/\, \precnsim \,/\, \prec] f(\bar{x}, s').$$

(c) For all $x', \bar{x} \in X$,

$$f_{\mathcal{U}}(x') - \mathbb{R}^p_{\overset{\succsim}{\approx}} \subseteq f_{\mathcal{U}}(\bar{x}) - \mathbb{R}^p_{\overset{\succsim}{\approx}} \Rightarrow \sup_{s \in \mathcal{U}} f_i(x', s) \leqslant \sup_{s' \in \mathcal{U}} f_i(\bar{x}, s'),$$

for all $i \in I_p$.

(d) If $\max\limits_{s \in \mathcal{U}} f_i(x, s)$ exists, for all $x \in X$ and $i \in I_p$, then for all $x', \bar{x} \in X$,

$$f_{\mathcal{U}}(x') - \mathbb{R}^p_{\overset{\succsim}{\approx}} \subseteq f_{\mathcal{U}}(\bar{x}) - \mathbb{R}^p_{\succ} \Rightarrow \max_{s \in \mathcal{U}} f_i(x', s) < \max_{s' \in \mathcal{U}} f_i(\bar{x}, s'),$$

for all $i \in I_p$.

Before we are going to closed this Chapter, we recall the result in Kalaï et al. [16] which will be used to prove the important proposition in Chapter III. To recall this fact, we remind the definition of ordering the values nonincreasingly which will be used throughout this thesis.

Definition 2.3.2. The sort function, $sort(\cdot) : \mathbb{R}^p \to \mathbb{R}^p$, is a function that reorders the component of each vector on $\mathbb{R}^p$ in a nonincreasing way. That is,

$$sort(y) = (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(p)}), \text{ for all } y \in \mathbb{R}^p \tag{2.3.1}$$

where σ is a permutation on I_p such that $y_{\sigma(1)} \geq y_{\sigma(2)} \geq \cdots \geq y_{\sigma(p)}$. In this case, we will write $sort(y) =: (sort_1(y), sort_2(y), \ldots, sort_p(y))$.

The following result was proposed by Kalaï et al. [16].

Lemma 2.3.3. [16] Given an uncertain single objective optimization problem and feasible solutions x, $y \in X$. If $f(x, s_j) \leqslant f(y, s_j)$, for each $s_j \in \mathcal{U}$, then

$$\hat{c}_j(x) \leqslant \hat{c}_j(y), \text{ for all } j \in I_q,$$

where $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ and the function $\hat{c}$ is defined as in (3.1.2).

CHAPTER III

THE LEXICOGRAPHIC TOLERABLE ROBUSTNESS CONCEPT

In this chapter, we introduce a new robust solution concept for uncertain multicriteria optimization problems and consider the important properties of the proposed concept. And, we present the method for finding the proposed solution. Some implementation of the solution concept of a problem of water resource management are shown and discussed.

3.1 Lexicographic robust solutions with respect to the tolerance threshold

In this section, we introduce the concept of lexicographic tolerable robust solution for an uncertain multicriteria optimization problem.

From now on, we let $\mathcal{U} = \{s_1, s_2, \ldots, s_q\}$ be the finite set of possible scenarios and $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^p$ be the considered vector-valued function. For each $x \in \mathbb{R}^n$ and for each $i \in I_p$, we put

$$c^{(i)}(x) := (f_i(x, s_1), f_i(x, s_2), \ldots, f_i(x, s_q)), \tag{3.1.1}$$

where $f_i(x, s_j)$ is the value of i^{th} component of the objective function for a feasible solution x under scenario s_j, for all $j \in I_q$. Subsequently, we put

$$\hat{c}^{(i)}(x) := \left(sort_1(c^{(i)}(x)), sort_2(c^{(i)}(x)), \ldots, sort_q(c^{(i)}(x))\right), \tag{3.1.2}$$

for each $i \in I_p$ and $x \in \mathbb{R}^n$. The notation $\hat{c}^{(i)}(x)$ is used to stand for the sorted vector of a vector $c^{(i)}(x)$. For the sake of simply, here we will write

$$\hat{c}^{(i)}(x) =: \left(\hat{c}_1^{(i)}(x), \hat{c}_2^{(i)}(x), \ldots, \hat{c}_q^{(i)}(x)\right), \tag{3.1.3}$$

for each $i \in I_p$. Accordingly, for each $j \in I_q$ and $x \in \mathbb{R}^n$, based on the above notations (3.1.1)-(3.1.3), the worst performance vector can be determined as follows:

$$worst_j(f(x,\mathcal{U})) := \left(\hat{c}_j^{(1)}(x), \hat{c}_j^{(2)}(x), \ldots, \hat{c}_j^{(p)}(x)\right). \tag{3.1.4}$$

Now we will introduce the concept of lexicographic robust solutions with respect to a tolerance threshold set for the considered uncertain multicriteria optimization problem. To do this, we start by introducing the notation $\inf\limits_{\text{with lex}} A$ which is used to stand for the infimum of a set A in $\mathbb{R}^p$ with respect to lexicographic order. That is, for $A \subseteq \mathbb{R}^p$, we let

$$x^{\inf} := \inf_{\text{with lex}} A \text{ if } x^{\inf} \leq_{lex} x, \text{ for all } x \in A$$

where $x^{\inf} \in \mathbb{R}^p$ and the notation $\leq_{lex}$ is defined as in Chapter II.

Here, the concept of reference point is presented.

Definition 3.1.1. The vector $\left(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*\right) =: \hat{c}^* \in \mathbb{R}^{p\times q}$ is called the *reference point* of the problem $\mathcal{MP}(\mathcal{U})$ if

$$\hat{c}_j^* = \inf_{\text{with lex}} \left\{ worst_j(f(x,\mathcal{U})) | x \in X \right\},$$

for each $j \in I_q$.

We now present the solution concept of lexicographic robust solutions with respect to a tolerance threshold for uncertain multicriteria optimization problems.

Definition 3.1.2. Let $\mathcal{MP}(\mathcal{U})$ be an uncertain multicriteria optimization problem with the reference point $\left(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*\right) =: \hat{c}^* \in \mathbb{R}^{p\times q}$. For each $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q) \in [0, \infty)^{p\times q}$, the *set of lexicographic tolerable robust solutions with respect to the tolerance threshold* α, which will be denoted by $LRS(\alpha)$, is

$$LRS(\alpha) := \bigcap_{j=1}^{q} A_j^{\alpha_j},$$

where $A_j^{\alpha_j} := \left\{x \in X | worst_j(f(x,\mathcal{U})) \in \left(\hat{c}_j^* + \alpha_j\right) - \mathbb{R}^p_{\succsim}\right\}$ for each $j \in I_q$.

Remark 3.1.3. (i) When $p = 1$, that is we are concerned with an uncertain single objective optimization problem, the concept of lexicographic tolerable robust solution in Definition 3.1.2 is identical to the lexicographic $\alpha-$robust solution which was introduced by Kalaï et al. [16]. In [16], for each $\alpha \in \mathbb{R}$, a set $A(X, \alpha)$ is called a set of lexicographic $\alpha-$robust solutions if

$$\begin{aligned} A(X,\alpha) &:= \bigcap_{j=1}^{q} A_j^{\alpha} \qquad (3.1.5) \\ &= \bigcap_{j=1}^{q} \{x \in X | \hat{c}_j(x) - \hat{c}_j^* \leqslant \alpha\}. \end{aligned}$$

(ii) When $|\mathcal{U}| = 1$, this means that we are dealing with deterministic multicriteria optimization problems. If α is the zero vector, then the solution concept in Definition 3.1.2 is nothing but the lexicographic solution concept in Definition 2.1.5 which we mentioned in Chapter II. Notice that there is a concept so-called TOPSIS as defined in Definition 2.1.8 that also involved the concept of the reference point (which is called the ideal point in Definition 2.1.8). However, the TOPSIS method and the lexicographic tolerable robust solution method do have significant differences in computation of the reference point. Indeed, according to the lexicographic tolerable robust solution concept, the reference point is derived by using the lexicographic order relation in comparing the vector in the image space. While, the TOPSIS method, the reference point will be computed by considering each respective component of the objective function separately by regardless the priority levels in the objective function.

3.2 Properties of solution set

Now the important properties of the solution set of lexicographic tolerable robust solutions will be studied and interpreted. We begin with the following fact that is immediately followed from the Lemma 2.3.3.

Proposition 3.2.1. Given $\mathcal{MP}(\mathcal{U})$ be an uncertain multicriteria optimization problem with the uncertainty set $\mathcal{U} = \{s_1, s_2, \ldots, s_q\}$. If x and y are feasible solutions in X and satisfy the relation

$$c^{(i)}(x) \precsim c^{(i)}(y), \text{ for all } i \in I_p, \tag{3.2.1}$$

then

$$worst_j(f(x,\mathcal{U})) \precsim worst_j(f(y,\mathcal{U})), \text{ for all } j \in I_q.$$

Proof. From the relation (3.2.1), by applying Lemma 2.3.3 to each value objective function f_i, we have

$$\hat{c}^{(i)}(x) \precsim \hat{c}^{(i)}(y),$$

for all $i \in I_p$. This immediately implies that

$$worst_j(f(x,\mathcal{U})) \precsim worst_j(f(y,\mathcal{U})), \text{ for all } j \in I_q.$$

□

Property 3.2.2. [Dominance] Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_q) \in [0, \infty)^{p\times q}$ and $x \in LRS(\alpha)$. If $y \in X$ satisfies

$$c^{(i)}(y) \precsim c^{(i)}(x), \text{ for all } i \in I_p, \tag{3.2.2}$$

then $y \in LRS(\alpha)$.

Proof. The proof is directly followed from Proposition 3.2.1 and Definition 3.1.2. □

Interpretation 3.2.3. The Property 3.2.2 stipulates that, if there is another feasible solution in a feasible set X which dominates a robust solution under all scenarios, then it must be a robust solution.

In order to proof the non preference property, we present the following fact.

Lemma 3.2.4. Let $X \subseteq \mathbb{R}^n$ and $x \in X$. If $x \precsim \inf_{\text{with lex}} X$, then $x = \inf_{\text{with lex}} X$.

Proof. For sake of simplicity, we write $y^* := \inf_{\text{with lex}} X$. Notice that from y^* is the infimum with respect to lexicographic order relation and $x \in X$, it imply that $y^* \leq_{lex} x$. Suppose that $x \neq y^*$. Then, there exists at least one $k \in I_n$ such that $x_k \neq y_k^*$. Defining $m := \min\{k | x_k \neq y_k^*\}$. From $x \precsim y^*$, it would follow that

$$x_i = y_i^*, \text{ for all } i = 1, 2, \dots, m-1 \text{ and } x_m < y_m^*.$$

This implies that $x <_{lex} y^*$. This leads to a contradiction with the assumption that y^* being the infimum. Hence, we can conclude that $x = y^*$. □

Next, we will consider the non preference property for the set $LRS(\alpha)$. To do so, we will consider the following binary relation with respect to a vector α under cone $\mathbb{R}^p_{\geqq}$. Let z and z' be vectors in $\mathbb{R}^p$, and $\alpha \in \mathbb{R}^p_{\geqq}$, the relation $\leqq^{\alpha}_{\mathbb{R}^p_{\geqq}}$ on $\mathbb{R}^p_{\geqq}$ is defined as follows:

$$z \leqq^{\alpha}_{\mathbb{R}^p_{\geqq}} z' \Leftrightarrow z' - \alpha \in z + \mathbb{R}^p_{\geqq}.$$

Proposition 3.2.5. **[Non Preference]** Let $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_q) \in [0, \infty)^{p \times q}$. If $x \in LRS(\alpha)$ and $z \notin LRS(\alpha)$, then for each $j \in I_q$, we have

$$worst_j(f(z,\mathcal{U})) \quad \nleqq^{\alpha_j}_{\mathbb{R}^p_{\geqq}} \quad worst_j(f(x,\mathcal{U})). \tag{3.2.3}$$

Proof. Let $j \in I_q$. Clearly, if $worst_j(f(z,\mathcal{U}))$ and $worst_j(f(x,\mathcal{U}))$ are not comparable, then the conclusion is obtained.

Now, we consider the case that $worst_j(f(z,\mathcal{U}))$ and $worst_j(f(x,\mathcal{U}))$ are comparable. Supposing on the contrary that

$$worst_j(f(z,\mathcal{U})) \leqq^{\alpha_j}_{\mathbb{R}^p_{\geqq}} worst_j(f(x,\mathcal{U})).$$

Thus, by the definition of $\leqq^{\alpha_j}_{\mathbb{R}^p_{\geqq}}$, we would get

$$worst_j(f(z,\mathcal{U})) \preccurlyeq worst_j(f(x,\mathcal{U})) - \alpha_j \text{ and } worst_j(f(z,\mathcal{U})) \neq worst_j(f(x,\mathcal{U}) - \alpha_j.$$

Subsequently, since $x \in LRS(\alpha)$, we see that

$$worst_j(f(z,\mathcal{U})) \preccurlyeq worst_j(f(x,\mathcal{U})) - \alpha_j \precapprox (\hat{c}_j^* + \alpha_j) - \alpha_j = \hat{c}_j^*. \tag{3.2.4}$$

Then, from the equation (3.2.4) together with the definition of the reference point $\hat{c}_j^*$ and Lemma 3.2.4, we could get

$$worst_j(f(z,\mathcal{U})) = \hat{c}_j^*. \tag{3.2.5}$$

In view of (3.2.4) and (3.2.5), we obtain that

$$worst_j(f(z,\mathcal{U})) = worst_j(f(x,\mathcal{U})) - \alpha_j,$$

which leads to a contradiction with the assumption that

$$worst_j(f(z,\mathcal{U})) \neq worst_j(f(x,\mathcal{U}) - \alpha_j.$$

Therefore,

$$worst_j(f(z,\mathcal{U})) \nleq^{\alpha_j}_{\mathbb{R}^p_{\geqslant}} worst_j(f(x,\mathcal{U})) \text{ for all } j \in I_q,$$

this completes the proof. □

Interpretation 3.2.6. The non preference property by Proposition 3.2.5 means that there is no preferable solution being outside the set $LRS(\alpha)$ with respect to the operator $worst_j(f(\cdot,\mathcal{U}))$, for each $j \in I_q$ and tolerance threshold $\alpha \in \mathbb{R}^p_{\geqslant}$.

The following proposition will be concerned with the stability of the solution set $LRS(\alpha)$.

Proposition 3.2.7. [Stability] For any $x, x' \in LRS(\alpha)$ where $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q) \in [0,\infty)^{p\times q}$, we have

$$worst_j(f(x',\mathcal{U})) \nleq^{\alpha_j}_{\mathbb{R}^p_{\geqslant}} worst_j(f(x,\mathcal{U}))$$

and

$$worst_j(f(x,\mathcal{U})) \nleq^{\alpha_j}_{\mathbb{R}^p_{\geqslant}} worst_j(f(x',\mathcal{U}))$$

for all $j \in I_q$.

Proof. Let $j \in I_q$ be fixed. Suppose on the contrary that

$$worst_j(f(x',\mathcal{U})) \leqq^{\alpha_j}_{\mathbb{R}^p_{\succcurlyeq}} worst_j(f(x,\mathcal{U}))$$

or

$$worst_j(f(x,\mathcal{U})) \leqq^{\alpha_j}_{\mathbb{R}^p_{\succcurlyeq}} worst_j(f(x',\mathcal{U})).$$

We may assume that $worst_j(f(x',\mathcal{U})) \leqq^{\alpha_j}_{\mathbb{R}^p_{\succcurlyeq}} worst_j(f(x,\mathcal{U}))$. From the definition of notation $\leqq^{\alpha_j}_{\mathbb{R}^p_{\succcurlyeq}}$, we see that

$$worst_j(f(x',\mathcal{U})) \preccurlyeq worst_j(f(x,\mathcal{U})) - \alpha_j.$$

Consequently, since $x \in LRS(\alpha)$, we get that

$$worst_j(f(x',\mathcal{U})) \preccurlyeq worst_j(f(x,\mathcal{U})) - \alpha_j \precsim (\hat{c}^*_j + \alpha_j) - \alpha_j = \hat{c}^*_j. \tag{3.2.6}$$

But (3.2.6) means that $worst_j(f(x',\mathcal{U}))$ is less than or equal to $\hat{c}^*_j$ in every component and there is at least one component of $worst_j(f(x',\mathcal{U}))$ which is strictly less than $\hat{c}^*_j$. This is a contradiction to the definition of $\hat{c}^*_j$ being the infimum of set $\{worst_j(f(x,\mathcal{U}))|x \in X\}$. Therefore, we have

$$worst_j(f(x',\mathcal{U})) \nleqq^{\alpha_j}_{\mathbb{R}^p_{\succcurlyeq}} worst_j(f(x,\mathcal{U})).$$

We can obtain the conclusion for the case $worst_j(f(x,\mathcal{U})) \leqq^{\alpha_j}_{\mathbb{R}^p_{\succcurlyeq}} worst_j(f(x',\mathcal{U}))$ by following anologeously the proof of case $worst_j(f(x',\mathcal{U})) \leqq^{\alpha_j}_{\mathbb{R}^p_{\succcurlyeq}} worst_j(f(x,\mathcal{U}))$. □

Interpretation 3.2.8. Proposition 3.2.7 shows that there is no preferable solution among elements in the set $LRS(\alpha)$ via considering the preference defined by the order relation $\leqq^{\alpha}_{\mathbb{R}^p_{\succcurlyeq}}$ on $\mathbb{R}^p_{\succcurlyeq}$.

3.3 The solution method

In this section, we are going to introduce the method which will be used for finding the proposed solutions.

3.3.1 The Nonemptyness of solution set

We begin this subsection with results on the tolerance threshold, results have been obtained to help decision makers in finding acceptable threshold to guarantee the nonemptyness of the solution set. In doing so, for the sake of simplicity, we use the following notations:

$$\max(x) := \max\{x_1, x_2, \ldots, x_n\},$$

and

$$x + \varepsilon := (x_1 + \varepsilon, x_2 + \varepsilon, \ldots, x_n + \varepsilon),$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and $\varepsilon \in \mathbb{R}$.

Next, we provides a threshold vector α such that the solution set $LRS(\alpha)$ is nonempty. Before we are going to the Theorem of Nonemptyness, we show an important tool.

Proposition 3.3.1. Let $\mathcal{U} = \{s_1, s_2, \ldots, s_q\}$ and $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ be a single objective function such that $f(\cdot, s_j) : \mathbb{R}^n \to \mathbb{R}$ is lower-semicontinuous on X, for each $j \in I_q$. Then, for each $j \in I_q$, the function $\hat{c}_j(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is lower-semicontinuous on X.

Proof. We will prove by induction. The result is true for the case $q = 2$, since $\hat{c}_1(\cdot) = \max\{f(\cdot, s_1), f(\cdot, s_2)\}$, and $\hat{c}_2(\cdot) = \min\{f(\cdot, s_1), f(\cdot, s_2)\}$.

Next, we assume that $f(\cdot, s_1), f(\cdot, s_2), \ldots, f(\cdot, s_k)$ are also lower-semicontinuous on $\mathbb{R}^n$ such that their corresponding sorting functions, $\hat{c}_1(\cdot), \hat{c}_2(\cdot), \ldots, \hat{c}_k(\cdot)$, are also lower-semicontinuous. Now, let $f(\cdot, s_{k+1})$ be a lower-semicontinuous function on $\mathbb{R}^n$.

Let us define the function $g_i : \mathbb{R}^n \to \mathbb{R}$ by

$$g_1(\cdot) = \max\{\hat{c}_1(\cdot), f(\cdot, s_{k+1})\} \text{ and } g_i(\cdot) = \max\{\hat{c}_i(\cdot), \min\{\hat{c}_{i-1}(\cdot), f(\cdot, s_{k+1})\}\},$$

for each $i \in \{2, 3, \ldots, k\}$ and

$$g_{k+1}(\cdot) = \min\{\hat{c}_k(\cdot), f(\cdot, s_{k+1})\}.$$

Observe that we have

$$g_1(x) \geq g_2(x) \geq \cdots \geq g_k(x) \geq g_{k+1}(x),$$

for all $x \in \mathbb{R}^n$. This means $\{g_1(\cdot), g_2(\cdot), \ldots, g_k(\cdot), g_{k+1}(\cdot)\}$ is the set of sort functions for $f(\cdot, s_1), f(\cdot, s_2), \ldots, f(\cdot, s_k), f(\cdot, s_{k+1})$. Moreover, from the induction hypothesis together with the property of lower-semicontinuous of $f(\cdot, s_{k+1})$, we have g_i is a lower-semicontinuous function for all $i \in \{1, 2, \ldots, k, k+1\}$. This completes the proof. □

Proposition 3.3.2. Let X be a feasible set and $\mathcal{MP}(\mathcal{U})$ an uncertain multicriteria optimization problem with the corresponding reference point $\left(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*\right) =: \hat{c}^* \in \mathbb{R}^{p \times q}$. Let $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q) \in \mathbb{R}^{p \times q}$ where $\alpha_j = (\alpha^{\inf}, \alpha^{\inf}, \ldots, \alpha^{\inf}) \in \mathbb{R}^p$, for all $j \in I_q$, such that

$$\alpha^{\inf} := \inf_{x \in X} \max\left(\Delta_x\right), \tag{3.3.1}$$

and $\Delta_x = \begin{bmatrix} worst_1(f(x,\mathcal{U})) - \hat{c}_1^* \\ worst_2(f(x,\mathcal{U})) - \hat{c}_2^* \\ \vdots \\ worst_q(f(x,\mathcal{U})) - \hat{c}_q^* \end{bmatrix} \in \mathbb{R}^{pq}.$

Then, for each $\varepsilon > 0$, we have

(i) $LRS(\alpha + \varepsilon) \neq \emptyset$, and

(ii) $LRS(\alpha - \varepsilon) = \emptyset$.

Proof. (i) Let $\varepsilon > 0$ be given. By the definition of $\alpha^{\inf}$, there exists $x_\varepsilon \in X$ such that

$$\alpha^{\inf} \leq \max(\Delta_{x_\varepsilon}) < \alpha^{\inf} + \varepsilon.$$

For each $j \in I_q$, we write $\hat{c}_j^* = \left(\hat{c}_j^{*(1)}, \hat{c}_j^{*(2)}, \ldots, \hat{c}_j^{*(p)}\right)$. It follows that,

$$\max_{i \in I_p} \left\{ \hat{c}_j^{(i)}(x_\varepsilon) - \hat{c}_j^{*(i)} \right\} < \alpha^{\inf} + \varepsilon,$$

for each $j \in I_q$. Subsequently, for each $j \in I_q$, we have

$$\hat{c}_j^{(i)}(x_\varepsilon) - \hat{c}_j^{*(i)} < \alpha^{\inf} + \varepsilon, \text{ for all } i \in I_p.$$

This implies that,

$$worst_j(f(x_\varepsilon, \mathcal{U})) \precsim \hat{c}_j^* + (\alpha_j + \varepsilon), \text{ for all } j \in I_q.$$

This shows that, $x_\varepsilon \in LRS(\alpha + \varepsilon)$, and the item (i) is proved.

(ii) Let $x \in X$ be arbitrary but fixed. By definition of $\alpha^{\inf}$, we know that

$$\alpha^{\inf} \leq \hat{c}_j^{(i)}(x) - \hat{c}_j^{*(i)}, \text{ for some } j \in I_q \text{ and } i \in I_p.$$

Thus, for each $\varepsilon > 0$, we must have

$$\alpha^{\inf} - \varepsilon < \hat{c}_j^{(i)}(x) - \hat{c}_j^{*(i)},$$

for some $j \in I_q$ and $i \in I_p$. This implies that $x \notin LRS(\alpha - \varepsilon)$. Since x is an arbitary element of X, we can conclude that the item (ii) is proved. □

Remark 3.3.3. If we define a function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$h(x) := \max(\Delta_x), \text{ for each } x \in \mathbb{R}^n,$$

then under the assumption $f_i(\cdot, s_1), f_i(\cdot, s_2), \ldots, f_i(\cdot, s_q)$ are continuous functions on X, for all $i \in I_p$, we can show that h is a continuous function on X. Indeed, for each fixed $j \in I_q$, let us consider a function $g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ which is defined by

$$g_j(x) = \max_{i \in I_p}\{\hat{c}_j^{(i)}(x) - c_j^*\}, \text{ for each } x \in \mathbb{R}^n.$$

Note that, by applying the Proposition 3.3.1, we have $\hat{c}_j^{(i)}(\cdot)$ is continuous on X, for each $i \in I_p$. This implies that g_j is a continuous function. Thus, under the continuity assumption on f_i, we may observe that the compactness property of the feasible set X is a sufficient condition for the well-defindness of the vector $\alpha \in \mathbb{R}^{pq}$, when we are in the situation that X is an in finite set.

The following theorem gives the sufficient conditions for the nonemptyness of solution set.

Theorem 3.3.4. **[Nonemptyness]** Let $\mathcal{MP}(\mathcal{U})$ be an uncertain multicriteria optimization problem with the corresponding reference point $(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*) := \hat{c}^* \in \mathbb{R}^{p\times q}$, and $f_i(\cdot, s_1), f_i(\cdot, s_2), \ldots, f_i(\cdot, s_q)$ be lower-semicontinuous functions on X, for all $i \in I_p$. Let $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q) \in \mathbb{R}^{p\times q}$ where $\alpha_j = (\alpha^{\inf}, \alpha^{\inf}, \ldots, \alpha^{\inf}) \in \mathbb{R}^p$, for all $j \in I_q$, such that a threshold value $\alpha^{\inf}$ is defined as (3.3.1). If X is a compact set then $LRS(\alpha)$ is nonempty.

Proof. Let $n \in \mathbb{N}$ be fixed. By choosing a threshold valued $\alpha^{\inf}$ as (3.3.1), we can find $x^n \in X$ such that

$$\alpha^{\inf} \leq \max(\Delta_{x^n}) < \alpha^{\inf} + \frac{1}{n}.$$

For each $j \in I_q$, we write $\hat{c}_j^* = \left(\hat{c}_j^{*(1)}, \hat{c}_j^{*(2)}, \ldots, \hat{c}_j^{*(p)}\right)$. It follows that,

$$\max_{i\in I_p}\left\{\ \hat{c}_j^{(i)}(x^n) - \hat{c}_j^{*(i)}\right\} < \alpha^{\inf} + \frac{1}{n},$$

for each $j \in I_q$. Accordingly, for each $j \in I_q$, we have

$$\hat{c}_j^{(i)}(x^n) - \hat{c}_j^{*(i)} < \alpha^{\inf} + \frac{1}{n}, \text{ for all } i \in I_p. \tag{3.3.2}$$

This means that,

$$worst_j(f(x^n,\mathcal{U})) \lessapprox \hat{c}_j^* + (\alpha_j + \frac{1}{n}), \text{ for all } j \in I_q.$$

It follows that,

$$x^n \in LRS(\alpha^{\inf} + \frac{1}{n}), \text{ for all } n \in \mathbb{N}.$$

Moreover, since X is compact and $\{x^n\} \subseteq X$, we let $\tilde{x} \in X$ and a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ be such that $x^{n_k} \to \tilde{x}$, as $k \to \infty$.

Since, for each $i \in I_p$, we have $f_i(\cdot, s_1), f_i(\cdot, s_2), \ldots, f_i(\cdot, s_q)$ are lower-semicontinuous, we know that $\hat{c}_j^{(i)}(\cdot)$ is also lower-semicontinuous function, for each $j \in I_q$. These imply,

$$\hat{c}_j^{(i)}(x^{n_k}) \to \hat{c}_j^{(i)}(\tilde{x}) \text{ as } k \to \infty, \tag{3.3.3}$$

for all $i \in I_p$ and $j \in I_q$. Using this one together with the continuity of maximum function, in view of (3.3.2), we have

$$\max_{j \in I_q} \max_{i \in I_p} \left\{ \hat{c}_j^{(i)}(x^{n_k}) - \hat{c}_j^{*(i)} \right\} \to \alpha^{\inf}, \text{ as } k \to \infty.$$

Thus by (3.3.3), we obtain that

$$\max_{j \in I_q} \max_{i \in I_p} \left\{ \hat{c}_j^{(i)}(\tilde{x}) - \hat{c}_j^{*(i)} \right\} = \alpha^{\inf}.$$

This guarantees that $\tilde{x} \in LRS(\alpha)$. This completes the proof. □

3.3.2 The Algorithm for finding solution set

By considering the Proposition 3.3.2 and Theorem 3.3.4, one can see that the tolerance threshold which is defined by (3.3.1) will be used to compute the best choice among the feasible solutions for the solution concept in Definition 3.1.2. In other words, the solution set due to Definition 3.1.2, for the considered uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$, is the set $LRS(\alpha)$ when α is computed by (3.3.1). The following Theorem 3.3.5 will lead to a method for computing an element in the such set $LRS(\alpha)$.

Theorem 3.3.5. Let $\mathcal{MP}(\mathcal{U})$ be an uncertain multicriteria optimization problem with the corresponding reference point $(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*) := \hat{c}^* \in \mathbb{R}^{p \times q}$, where $\hat{c}_j^* := \left(\hat{c}_j^{*(1)}, \hat{c}_j^{*(2)}, \ldots, \hat{c}_j^{*(p)}\right) \in \mathbb{R}^p$, for all $j \in I_q$. Let $\alpha := (\alpha_1, \ldots, \alpha_q) \in [0, \infty)^{p \times q}$ be such that $\alpha_j := \left(\alpha_j^{(1)}, \alpha_j^{(2)}, \ldots, \alpha_j^{(p)}\right) \in \mathbb{R}^p$ for all $j \in I_q$. Then, we have

$$\bigcap_{(i,j) \in I_p \times I_q} L_{(i,j)} \subseteq LRS(\alpha),$$

where $L_{(i,j)} = \left\{x \in X | \hat{c}_j^{(i)}(x) \leqslant \hat{c}_j^{*(i)} + \alpha_j^{(i)}\right\}$ for all $i \in I_p$ and $j \in I_q$.

Proof. Let $x \in \bigcap_{(i,j)\in I_p \times I_q} L_{(i,j)}$. This means that,

$$x \in \{z \in X | \hat{c}_j^{(i)}(z) \leqslant \hat{c}_j^{*(i)} + \alpha_j^{(i)}\}, \text{ for all } i \in I_p \text{ and } j \in I_q.$$

This implies that,

$$x \in \{z \in X | worst_j(f(z,\mathcal{U})) \lessapprox \hat{c}_j^* + \alpha_j\}, \text{ for all } j \in I_q.$$

Thus, it follows directly that $x \in LRS(\alpha)$ and the theorem is proved. □

Based on Theorem 3.3.5, we now suggest a method for finding a solution to the problem $\mathcal{MP}(\mathcal{U})$ in the set $LRS(\alpha)$.

Algorithm 1: Finding a lexicographic tolerable robust solution of $\mathcal{MP}(\mathcal{U})$

Input: Uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$.

Step.1: For each fixed $j \in I_q$, find the reference point $\hat{c}_j^*$.

Step.2: Compute a tolerance threshold valued $\alpha^{\inf}$ of the problem $\mathcal{MP}(\mathcal{U})$ as defined in the Equation (3.3.1) of Proposition 3.3.2.

Step.3: For each fixed $i \in I_p$ and $j \in I_q$, compute the level set $L_{(i,j)}$ by

$$L_{(i,j)} = \left\{x \in X | \hat{c}_j^{(i)}(x) \leqslant \hat{c}_j^{*(i)} + \alpha^{\inf}\right\}, \tag{3.3.4}$$

where $\hat{c}_j^* := \left(\hat{c}_j^{*(1)}, \hat{c}_j^{*(2)}, \ldots, \hat{c}_j^{*(p)}\right)$.

Step.4: Find an element x^* in the set

$$\bigcap_{(i,j)\in I_p \times I_q} L_{(i,j)}.$$

Output: x^* is an element of $LRS(\alpha)$, where $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q) \in \mathbb{R}^{p\times q}$, such that $\alpha_j = (\alpha^{\inf}, \alpha^{\inf}, \ldots, \alpha^{\inf}) \in \mathbb{R}^p$, for all $j \in I_q$.

Remark 3.3.6. (i) For each $j \in I_q$, the vector $\hat{c}_j^* \in \mathbb{R}^p$ is found by finding the value of lexicographic optimization of the deterministic multicriteria mapping $worst_j(f(\cdot,\mathcal{U})) : \mathbb{R}^n \to \mathbb{R}^p$. For information on methods for finding the reference point as in Definition 3.1.1, one may see [18].

(ii) Observe that the computation of value $\alpha^{\inf}$ is finding the infimum of subset of real numbers. Thus, we can apply many elementary existing methods of finding this value.

(iii) Under the assumptions of $f_i(\cdot, s_j)$ being continuous for all $i \in I_p$ and $j \in I_q$, together with an assumption that the feasible set X is compact, by applying Proposition 3.3.1, we have $\hat{c}_j^{(i)}(\cdot)$ is also continuous for all $i \in I_p$ and $j \in I_q$. Thus, since $\hat{c}_j^{(i)}(\cdot)$ is continous, we have that the level set $L_{(i,j)}$ as defined by (3.3.4) is also a closed set. Thus, in order to finding a point in formulation (3.3.5) and complete Step 4, we can apply many existing algorithms, we refer the reader to [19, 20, 21].

3.3.3 Ranking of solution

In practice, the process of selecting a final solution for the considered problem usually involves multiple decision makers. Furthermore, there may occur the situation that some decision makers are not satisfied with the solution found by the lexicographic tolerable robust solution concept. Consequently, we may need to find more desirable solutions to offer those decision makers. Reasonably, in order to update the solution for fitting the preference or requirements of those decision makers, the monotonicity of the solution set is a vital property that the presented solution concept must satisfy. The following statement describes the monotonicity property of the solution set $LRS(\alpha)$.

Property 3.3.7. **[Monotonicity]** The set $LRS(\alpha)$ is monotonic in the tolerance threshold set. That is, for $\alpha := (\alpha_1, \ldots, \alpha_q), \beta := (\beta_1, \ldots, \beta_q) \in \mathbb{R}^{p\times q}$ such that $\alpha_j \precsim \beta_j$, for all $j \in I_q$, we have

$$LRS(\alpha) \subseteq LRS(\beta).$$

Proof. The proof directly follows from Definition 3.1.2. $\square$

Remark 3.3.8. Property 3.3.7 means that once the tolerance threshold set has been adjusted using small tolerance threshold values it will also function correctly with larger tolerance threshold values. In other words, a lexicographic robust solution correctly adjusted with low tolerance threshold values will remain a lexicographic robust solution even when the tolerance threshold values are high.

Continuing from above discussion, in order to update the solution sets, the ranking concept needs to be considered. Here, we consider the common natural idea for the ranking of different sets as we shall begin with computing the smallest tolerance threshold such that the set $LRS(\alpha)$ is nonempty (see Theorem 3.3.4) and define it to be a tolerance threshold set of the first ranking of solution set. After that, the next ranking of the solution set can be computed by removing all elements that belong to the first ranking of solution set from the feasible set. This mentioned idea is encouraged by the following Theorem 3.3.9, which we present in the suitation that the feasible solution set X is finite.

Theorem 3.3.9. Let X be a finite set and for each $m \in \{2, 3, \dots, q\}$, let α^m defined by

$$\alpha^m := \min_{x \in X \setminus LRS((\alpha^{m-1}, \dots, \alpha^{m-1}), \dots, (\alpha^{m-1}, \dots, \alpha^{m-1}))} \max(\Delta_x), \tag{3.3.5}$$

where $\alpha^1 := \min_{x \in X} \max(\Delta_x)$. Then, for any $\beta \in [\alpha^m, \alpha^{m+1})$, we have

$$LRS((\beta, \dots, \beta), \dots, (\beta, \dots, \beta)) = LRS((\alpha^m, \dots, \alpha^m), \dots, (\alpha^m, \dots, \alpha^m)).$$

Proof. By the monotonicity of a solution set, the "$\supseteq$" inclusion is obvious. So, we need to show that

$$LRS((\beta, \dots, \beta), \dots, (\beta, \dots, \beta)) \subseteq LRS((\alpha^m, \dots, \alpha^m), \dots, (\alpha^m, \dots, \alpha^m)).$$

Suppose on the contrary, there is $\bar{x} \in LRS((\beta,\dots,\beta),\dots,(\beta,\dots,\beta))$, but

$$\bar{x} \notin LRS((\alpha^m,\dots,\alpha^m),\dots,(\alpha^m,\dots,\alpha^m)).$$

It means that $\bar{x} \in X \backslash LRS((\alpha^m,\dots,\alpha^m),\dots,(\alpha^m,\dots,\alpha^m))$ and by the definition of α^{m+1},

$$\alpha^{m+1} \leqslant \max(\Delta_{\bar{x}}). \tag{3.3.6}$$

This implies that, there are $j_0 \in I_q$ and $i_0 \in I_p$ such that

$$\max(\Delta_{\bar{x}}) = \hat{c}_{j_0}^{(i_0)}(\bar{x}) - \hat{c}_{j_0}^{*(i_0)}.$$

Since $\bar{x} \in LRS((\beta,\dots,\beta),\dots,(\beta,\dots,\beta))$,

$$worst_{j_0}(f(\bar{x},\mathcal{U})) \in \hat{c}_{j_0}^{*} + (\beta,\beta,\dots,\beta) - \mathbb{R}^p_{\precapprox}.$$

By the definition of $\mathbb{R}^p_{\precapprox}$, it follows that,

$$worst_{j_0}(f(\bar{x},\mathcal{U})) \precapprox \hat{c}_{j_0}^{*} + (\beta,\beta,\dots,\beta).$$

This implies that, for any $i \in I_p$,

$$\hat{c}_{j_0}^{(i)}(\bar{x}) \leqslant \hat{c}_{j_0}^{*(i)} + \beta.$$

So, for fixed $i_0 \in I_p$,

$$\hat{c}_{j_0}^{(i_0)}(\bar{x}) \leqslant \hat{c}_{j_0}^{*(i_0)} + \beta. \tag{3.3.7}$$

From Equations (3.3.6) and (3.3.7), it follows that

$$\alpha^{m+1} \leqslant \hat{c}_{j_0}^{(i_0)}(\bar{x}) - \hat{c}_{j_0}^{*(i_0)} \leqslant \beta.$$

Which leads to a contradiction with the definition of β. Therefore, we obtain the inclusion and so

$$LRS((\beta,\ldots,\beta),\ldots,(\beta,\ldots,\beta)) = LRS((\alpha^m,\ldots,\alpha^m),\ldots,(\alpha^m,\ldots,\alpha^m)).$$

□

It should be noted that the lexicographic tolerable robust solution depending on the choice of tolerance threshold α. Theorem 3.3.9 provides a sufficient condition on how to choose the effective tolerance threshold for classifying the ranking on the solution set. The resulting solution set will remain the same set as the previous ranking of the solution set if a tolerance threshold does not reach to at least a value which was computed by (3.3.5). For more understanding on Theorem 3.3.9, we illustrate with the remark 3.5.2 in Section 3.5.

3.3.4 Refinement of the tolerance threshold

It is worth to remind that the lexicographic robust solutions sets depend on the considered tolerance threshold. Moreover, by Theorem 3.3.9, it has been asserted that there is no solution set that properly lies between the $LRS(\alpha^i)$ and $LRS(\alpha^{i+1})$ when these α^i are computed by the method presented in the formulation 3.3.5. Indeed, by choice of tolerance threshold α^i which is computed by Theorem 3.3.9 could be obtained new members in $LRS(\alpha^i)$ more than one element, one may wonder whether these new members are really in the same rank (we illustrate this observation with the remark 3.5.2 in Section 3.5). Here, we consider the idea to sharpen the ranking of the solution and the computation of tolerance threshold to determine a sub-rank among elements in the i^{th} ranking of the solution set is presented. The first sub-rank of the i^{th} ranking can be determined by computing the following formulation of the tolerance threshold:

$$\alpha^{i_1} := \inf_{\text{with lex}} \left\{ (\alpha_1^{i_1}(x), \alpha_2^{i_1}(x), \ldots, \alpha_q^{i_1}(x)) \in \mathbb{R}^{p\times q} | x \in LRS(\alpha^i) \setminus LRS(\alpha^{i-1}) \right\},$$

where

$$\alpha_j^{i_1}(x) = \Big(\max\{\hat{c}_j^{(1)}(x) - \hat{c}_j^{*(1)}, 0\}, \max\{\hat{c}_j^{(2)}(x) - \hat{c}_j^{*(2)}, 0\}, \ldots, \max\{\hat{c}_j^{(p)}(x) - \hat{c}_j^{*(p)}, 0\}\Big) \in \mathbb{R}^p,$$

and $\hat{c}_j^* := \Big(\hat{c}_j^{*(1)}, \hat{c}_j^{*(2)}, \ldots, \hat{c}_j^{*(p)}\Big) \in \mathbb{R}^p$, for all $j \in I_q$. The resulting solution set $LRS(\alpha^{i_1})$ corresponding to a tolerance threshold α^{i_1} is considered as a robust solution in the first sub-rank of the i^{th} ranking of the solution set.

The process of computing a tolerance threshold to determine the second sub-rank of the i^{th} ranking of the solution set will be continued if the remaining solution set $LRS(\alpha^i) \setminus \{LRS(\alpha^{i_1}) \cup LRS(\alpha^{i-1})\}$ is nonempty. Consequently, the second sub-rank of the i^{th} ranking of the solution set can be determined by computing the following formulation of the tolerance threshold:

$$\alpha^{i_2} := \inf_{\text{with lex}} \Big\{(\alpha_1^{i_2}(x), \alpha_2^{i_2}(x), \ldots, \alpha_q^{i_2}(x)) \in \mathbb{R}^{p\times q} | x \in LRS(\alpha^i) \setminus \{LRS(\alpha^{i-1}) \cup LRS(\alpha^{i_1})\}\Big\},$$

where

$$\alpha_j^{i_2}(x) = \Big(\max\{\hat{c}_j^{(1)}(x) - \hat{c}_j^{*(1)}, 0\}, \max\{\hat{c}_j^{(2)}(x) - \hat{c}_j^{*(2)}, 0\}, \ldots, \max\{\hat{c}_j^{(p)}(x) - \hat{c}_j^{*(p)}, 0\}\Big) \in \mathbb{R}^p,$$

where $j \in I_q$. The resulting solution set $LRS(\alpha^{i_2})$ corresponding to the tolerance threshold α^{i_2} is considered as a robust solution in the second sub-rank of the i^{th} ranking of the solution set.

We will continue this process of computing the third sub-rank if $LRS(\alpha^i) \setminus \{LRS(\alpha^{i_1}) \cup LRS(\alpha^{i_2}) \cup LRS(\alpha^{i-1})\}$ is nonempty. The third sub-rank of the i^{th} ranking of the solution set is determined by the following tolerance threshold:

$$\begin{aligned}\alpha^{i_3} := \inf_{\text{with lex}} \Big\{ \; & (\alpha_1^{i_3}(x), \alpha_2^{i_3}(x), \ldots, \alpha_q^{i_3}(x)) \in \mathbb{R}^{p\times q} | \\ & x \in LRS(\alpha^i) \setminus \{LRS(\alpha^{i-1}) \cup LRS(\alpha^{i_1}) \cup LRS(\alpha^{i_2})\}\Big\},\end{aligned}$$

where

$$\alpha_j^{i_3}(x) = \Big(\max\{\hat{c}_j^{(1)}(x) - \hat{c}_j^{*(1)}, 0\}, \max\{\hat{c}_j^{(2)}(x) - \hat{c}_j^{*(2)}, 0\}, \ldots, \max\{\hat{c}_j^{(p)}(x) - \hat{c}_j^{*(p)}, 0\}\Big) \in \mathbb{R}^p,$$

where $j \in I_q$. We do continue the process of computing the next sub-rank until there is $m \in \mathbb{N}$ such that

$$LRS(\alpha^i) = LRS(\alpha^{i_1}) \cup LRS(\alpha^{i_2}) \cup \cdots \cup LRS(\alpha^{i_m}).$$

In general, the above formulation of computing the tolerance threshold to determine the k^{th} sub-rank of the i^{th} ranking can be expressed as follows:

$$\alpha^{i_k} := \inf_{\text{with lex}} \Big\{ (\alpha_1^{i_k}(x), \alpha_2^{i_k}(x), \ldots, \alpha_q^{i_k}(x)) \in \mathbb{R}^{p\times q} | \\ x \in LRS(\alpha^i) \setminus \{LRS(\alpha^{i-1}) \cup LRS(\alpha^{i_1}) \cup \cdots \cup LRS(\alpha^{i_{k-1}})\} \Big\} \quad (3.3.8)$$

where

$$\alpha_j^{i_k}(x) = \Big(\max\{\hat{c}_j^{(1)}(x) - \hat{c}_j^{*(1)}, 0\}, \max\{\hat{c}_j^{(2)}(x) - \hat{c}_j^{*(2)}, 0\}, \ldots, \max\{\hat{c}_j^{(p)}(x) - \hat{c}_j^{*(p)}, 0\} \Big) \in \mathbb{R}^p,$$

where $k \in \mathbb{N}$ and $j \in I_q$.

It is noteworthy that there is a situation that even we refine the tolerance threshold by the formulation (3.3.8), the corresponding sub-rank of solution sets with respect to the updating tolerance threshold may not be singleton sets. The following example will provide an affirmative conclusion for this observation.

Example 3.3.10. Let $X = \{x^1, x^2, x^3\}$, and the vector-valued function f under two possible scenarios s_1 and s_2 of each feasible solution x^k be presented as Table 1. The sort function of each component function f_i and the jth worst performance vector of each feasible solution x^k are provided in Table 2.

According to Theorem 3.3.9, the first ranking of the solution set and the second ranking of the solution set are $LRS(\alpha^1) = \{x^1\}$, and $LRS(\alpha^2) = \{x^2, x^3\}$, where $\alpha^1 := ((0,0),(0,0))$ and $\alpha^2 := ((4,4),(4,4))$. To refine the tolerance threshold α^2, we can now apply the formulation (3.3.8) and so the tolerance threshold for determining the first sub-rank of the 2^{nd} ranking of the solution set is:

$$\alpha^{2_1} := ((1,0),(1,4)).$$

Notice that

$$worst_1(f(x^2,\mathcal{U})) \in (\hat{c}_1^* + (1,0)) + \mathbb{R}^2_{\gtrapprox} \text{ and } worst_2(f(x^2,\mathcal{U})) \in (\hat{c}_2^* + (1,4)) + \mathbb{R}^2_{\gtrapprox},$$

and

$$worst_1(f(x^3,\mathcal{U})) \in (\hat{c}_1^* + (1,0)) + \mathbb{R}^2_{\gtrapprox} \text{ and } worst_2(f(x^3,\mathcal{U})) \in (\hat{c}_2^* + (1,4)) + \mathbb{R}^2_{\gtrapprox}.$$

This mean that the first sub-rank of the 2^{nd} ranking of the solution set is the set $LRS(\alpha^{2_1}) = \{x^2, x^3\}$. Therefore, we can conclude that by using the tolerance threshold which is computed by the formulation (3.3.8), cannot guarantee the corresponding singleton solution set of the sub-rank.

	Objective Function			
Alternatives	$f_1(\cdot, s_1)$	$f_1(\cdot, s_2)$	$f_2(\cdot, s_1)$	$f_2(\cdot, s_2)$
x^1	5	6	11	2
x^2	7	6	8	6
x^3	6	7	6	7

Table 1: The objective function $f = (f_1, f_2)$ for each feasible solution x^k under all scenarios s_j of Example 3.3.10.

Alternatives	$\hat{c}^{(1)}(\cdot)$	$\hat{c}^{(2)}(\cdot)$	$worst_1(f(\cdot,\mathcal{U}))$	$worst_2(f(\cdot,\mathcal{U}))$
x^1	$(6,5)$	$(11,2)$	$(6,11)$	$(5,2)$
x^2	$(7,6)$	$(8,6)$	$(7,8)$	$(6,6)$
x^3	$(7,6)$	$(7,6)$	$(7,7)$	$(6,6)$
$\hat{c}_j^*$			$\hat{c}_1^* = (6,11)$	$\hat{c}_2^* = (5,2)$

Table 2: The function $\hat{c}^{(i)}(\cdot)$ and $worst_j(f(\cdot,\mathcal{U}))$ of Example 3.3.10

The observation from Example 3.3.10 is that even we refine the tolerance threshold by using the formulation (3.3.8), the corresponding solution set with respect to such tolerance threshold can sometime be not singleton set. Indeed, the robust solutions which belong to the k^{th} sub-rank of the i^{th} ranking of the solution set are indifferent because the quality of these robust solutions which are computed by the lexicographic tolerable robust solution concept are same, mean that the worst performance vectors of these robust solutions are located in an acceptable area corresponding to the tolerance threshold α^{i_k}.

Remark 3.3.11. Notice that by choice of α^i and the formulation (3.3.8), of computing k^{th} sub-rank of the i^{th} ranking, we can see that $\alpha^{i_k} \precsim \alpha^i$.

3.4 Lexicographic tolerable robust solution and the set-based minmax robust efficiency

In this section, we consider the links between the lexicographic tolerable robust solution and the set-based minmax robust efficiency, which was introduced by Ehrgott et al. [8]. The following result presents the technique for finding the set-based robust efficiency via the lexicographic tolerable robust solution idea.

Theorem 3.4.1. Let $\mathcal{MP}(\mathcal{U})$ be an uncertain multicriteria optimization problem with the reference point $(\hat{c}_1^*, \hat{c}_2^*, \dots, \hat{c}_q^*) =: \hat{c}^* \in \mathbb{R}^{p\times q}$ and 0 the zero vector in $\mathbb{R}^p$. If $\hat{x} \in A_1^0$ and A_1^0 is defined as in Definition 3.1.2, then $\hat{x}$ is a set-based robust weakly efficient solution for the problem $\mathcal{MP}(\mathcal{U})$.

Proof. Since $\hat{x} \in A_1^0$, we have that

$$worst_1(f(\hat{x},\mathcal{U})) \quad \precsim \quad \hat{c}_1^*. \tag{3.4.1}$$

By the definition of reference point $\hat{c}_1^*$ together with Lemma 3.2.4, it follows that

$$worst_1(f(\hat{x},\mathcal{U})) \quad = \quad \hat{c}_1^*. \tag{3.4.2}$$

Suppose that $\hat{x}$ is not a set-based weakly robust efficient solution. Then, there exists $x \in X \setminus \{\hat{x}\}$ such that

$$f_{\mathcal{U}}(x) \subseteq f_{\mathcal{U}}(\hat{x}) - \mathbb{R}^p_{\succ}. \tag{3.4.3}$$

By applying the items (a) and (d) of Lemma 2.3.1, we have that

$$\max_{s \in \mathcal{U}} f_i(x,s) < \max_{s \in \mathcal{U}} f_i(\hat{x},s), \text{ for all } i \in I_p. \tag{3.4.4}$$

By the definition of $\hat{c}_1^{(i)}(\cdot)$ and definition of $worst_1(f(\cdot,\mathcal{U}))$, it imply that

$$worst_1(f(x,\mathcal{U})) \preccurlyeq worst_1(f(\hat{x},\mathcal{U})) = \hat{c}_1^*. \tag{3.4.5}$$

This leads to a contradiction with $\hat{c}_1^*$ being the infimum of set $\{worst_1(f(x,\mathcal{U}))|x \in X\}$. Therefore $\hat{x}$ is a set-based robust weakly efficient solution for the problem $\mathcal{MP}(\mathcal{U})$. □

The following result provides a sufficient condition on a technique for finding the set-based robust strictly efficient solution for the problem $\mathcal{MP}(\mathcal{U})$.

Theorem 3.4.2. Let $\mathcal{MP}(\mathcal{U})$ be an uncertain multicriteria optimization problem together with reference point $\left(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*\right) =: \hat{c}^* \in \mathbb{R}^{p\times q}$. Let $\alpha_1 = (\alpha, \alpha, \ldots, \alpha) \in \mathbb{R}^p_{\gtrapprox}$. If $A_1^{\alpha_1} = \{\hat{x}\}$ and $A_1^{\alpha_1}$ is defined as in Definition 3.1.2, then $\hat{x}$ is a set-based minmax robust strictly efficient solution for the problem $\mathcal{MP}(\mathcal{U})$.

Proof. From $\hat{x} \in A_1^{\alpha_1}$, it follows that

$$worst_1(f(\hat{x},\mathcal{U})) \in (\hat{c}_1^* + \alpha_1) - \mathbb{R}^p_{\gtrapprox}.$$

This means that,

$$worst_1(f(\hat{x},\mathcal{U})) \lessapprox \hat{c}_1^* + \alpha_1. \tag{3.4.6}$$

For each $j \in I_q$, we write $\hat{c}_j^* = \left(\hat{c}_j^{*(1)}, \hat{c}_j^{*(2)}, \ldots, \hat{c}_j^{*(p)}\right)$. It follows that,

$$\hat{c}_1^{(i)}(\hat{x}) \leqslant \hat{c}_1^{*(i)} + \alpha, \text{ for all } i \in I_p.$$

Suppose on the contrary, that $\hat{x}$ is not set-based minmax robust strictly efficient for the problem $\mathcal{MP}(\mathcal{U})$. By the Definition 2.2.5, there is $x \in X \setminus \{\hat{x}\}$ such that

$$f_{\mathcal{U}}(x) \subseteq f_{\mathcal{U}}(\hat{x}) - \mathbb{R}^p_{\gtrapprox}.$$

By the items (a), (b) and (c) of Lamma 2.3.1, we obtain that

$$\max_{s\in\mathcal{U}} f_i(x,s) \;\leqslant\; \max_{s\in\mathcal{U}} f_i(\hat{x},s), \text{ for all } i \in I_p. \tag{3.4.7}$$

We note that for each $i \in I_p$, by the definition of $\hat{c}_1^{(i)}(\cdot)$, we have that

$$\hat{c}_1^{(i)}(x) \;=\; \max_{s\in\mathcal{U}} f_i(x,s), \text{ for all } x \in X. \tag{3.4.8}$$

From the equations (3.4.7) and (3.4.8), it follow that

$$worst_1(f(x,\mathcal{U})) \lesssim worst_1(f(\hat{x},\mathcal{U})). \tag{3.4.9}$$

Thus, from the equations (3.4.6) and (3.4.9), we must have that x is an another element in $A_1^{\alpha_1}$. This leads to a contradiction with the assumption that $\hat{x}$ being the unique element in $A_1^{\alpha_1}$. Therefore, $\hat{x}$ is a set-based minmax robust strictly efficient solution for the problem $\mathcal{MP}(\mathcal{U})$. □

Remark 3.4.3. To find a threshold α_1 that the set $A_1^{\alpha_1}$ is a singleton, is not so difficult. This is because, if we start with tolerance threshold α_1, then one can be refined to obtain the desirable singleton set $A_1^{\alpha_1^{new}}$, where the updating tolerance threshold α_1^{new} can be computed via the method which was presented in Section 3.3. Nevertheless, there is only one situation that even we refine the tolerance threshold α_1, the corresponding set $A_1^{\alpha_1^{new}}$ still not singleton, that is there are several feasible solutions providing the same worst performance vector $worst_1(f(\cdot,\mathcal{U}))$.

The following example shows a situation that $A_1^{\alpha_1}$ is not a singleton set for any choice of α_1.

Example 3.4.4. Let $X = \{x_1, x_2, x_3, x_4\}$ be the considered feasible set. The information about vector-valued function f for each feasible solution x_i estimated under two possible scenarios s_1 and s_2 are shown in Table 3. Consequently, the sort function $\hat{c}^{(i)}(\cdot)$ of each component function f_i and also the j^{th} worst performance vector of each feasible solution x_i are provided in Table 4. Thus, it follows that $A_1^{\alpha_1} = \{x_2, x_3, x_4\}$, for all $\alpha_1 := (\alpha, \alpha) \in \mathbb{R}^2_{\succsim}$.

Moreover, according to the Definition 2.2.5, we can check that the set-based minmax robust efficient solution set is $\{x_1, x_4\}$. This means, for any choice of α_1, the set $A_1^{\alpha_1}$ is not a subset of set-based minmax robust efficient solution set. Furthermore, by applying the method of computing the smallest tolerance threshold which can be guaranteed the nonemptyness of the lexicographic tolerable robust solution concept in Definition 3.1.2, we found that the such solution set is $LRS(\alpha^*) = \{x_3\}$ with $\alpha^* := ((1,1),(1,1))$. This shows that the solution sets of those related to lexicographic tolerable robust solution concept and set-based minmax robust efficiency solution concept can be (extremely) different.

Table 3: The objective function $f = (f_1, f_2)$ for each feasible solution x_k under all scenarios s_j of Example 3.4.4

	Objective Function			
Alternatives	$f_1(\cdot, s_1)$	$f_1(\cdot, s_2)$	$f_2(\cdot, s_1)$	$f_2(\cdot, s_2)$
x_1	4	11	14	4
x_2	10	10	11	7
x_3	10	5	11	4
x_4	10	6	7	11

Alternatives	$\hat{c}^{(1)}(\cdot)$	$\hat{c}^{(2)}(\cdot)$	$worst_1(f(\cdot,\mathcal{U}))$	$worst_2(f(\cdot,\mathcal{U}))$
x_1	$(11,4)$	$(14,4)$	$(11,14)$	$(4,4)$
x_2	$(10,10)$	$(11,7)$	$(10,11)$	$(10,7)$
x_3	$(10,5)$	$(11,4)$	$(10,11)$	$(5,4)$
x_4	$(10,6)$	$(11,7)$	$(10,11)$	$(6,7)$
$\hat{c}_j^*$			$\hat{c}_1^* = (10,11)$	$\hat{c}_2^* = (4,4)$

Table 4: The function $\hat{c}^{(i)}(\cdot)$ and $worst_j(f(\cdot,\mathcal{U}))$ of Example 3.4.4

3.5 Case study: A water resources management

In this study we consider data from [22] and solve the problem by using the lexicographic tolerlable robust solution concept.

3.5.1 Problem statement

The original problem of Water Resources Master Plan for Serbia (WRMS) is to find a suitable plan for balancing water demands and available water resources. This problem is concerned with the six feasible solutions and eight objectives as follows:

Decision factors:

- The need for municipal water supply (d_1)
- The need for industrial water supply (d_2)
- Irrigation needs (d_3)
- Hydropower generation (d_4)
- Flood protection (d_5)
- Water quality control (d_6)

Objectives:

- Regional political interest (f_1)
- Local interest (communities) (f_2)
- Negative effects on the resettlement of people (f_3)
- System reliability (f_4)
- Positive environmental effects (f_5)
- Positive effects of alternative plans on water quality (f_6)
- Total cost (f_7)
- Energy consumption (f_8)

The modelling techniques of feasible solution and the measurement of the objective function, we refer the readers to see more details in Chapter 10 of [22]. Here, six feasible solutions were created by considering the above specific factors in the planning process for the WRMS problem. Thus, the decision space is $X := \{x^1, x^2, x^3, x^4, x^5, x^6\} \subseteq \mathbb{R}^6$ where $x^k := (d_1^k, d_2^k, d_3^k, d_4^k, d_5^k, d_6^k)$ for each $k \in I_6$.

Looking at the above eight objectives, one can see that the five objectives f_1, f_2, f_4, f_5, and f_6, relate to positive outcomes that the group of decision makers naturally wants to maximize. Meanwhile, the three objectives f_3, f_7, and f_8 relate to negative outcomes that they naturally want to minimize. Notice that the first six objectives are qualitative, while the remaining two are quantitative. The quality level of the first six objectives are divided by the relative scale into five levels from being bad to being excellent as 1 to 5.

In the solution selection process, the preferences of the decision makers were collected through a set of public meetings. Since the decision makers were not able or willing to express their preferences, the planning team have to generate a number of different sets of weights to cover a broad range of decision-making positions in accordance with the relative importance of the various objectives. Since the generation of weight sets in the WRMS is obtained from ranges of decision maker's preferences, it means that the weight sets are imprecise data. Hence, these imprecise data can be seen as an uncertainty in the WRMS. So, it is reasonable to consider a robustness concept for the WRMS that is quite sensitive to preference changes of the decision makers. Six different weight sets had been presented in the WRMS, and here these six weight sets will be considered as scenarios. That is, the uncertainty set is:

$$\mathcal{U} := \{s_1, s_2, \ldots, s_6\} \subseteq \mathbb{R}^8.$$

Therefore, the WRMS problem is formulated as an uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$ where $\mathcal{MP}(\mathcal{U})$ is given as a family of $\{\mathcal{MP}(s_j)|s_j \in \mathcal{U}\}$ of deterministic multicriteria optimization problems:

$$(\mathcal{MP}(s_j)) \quad \min \quad f(x^k, s_j) \qquad (3.5.1)$$
$$\text{subject to } x^k \in X$$

where $f : X \times \mathcal{U} \to \mathbb{R}^8$, $X = \{x^1, x^2, x^3, x^4, x^5, x^6\}$, and $\mathcal{U} = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. The primary data of the outcome for each feasible solution x^k in the WRMS over all scenarios are shown in Table 6.

The Ordered Objective Groups	The $LRS(\alpha)$ Solution
(G_1, G_2, G_3)	x^3
(G_1, G_3, G_2)	x^5
(G_2, G_1, G_3)	x^6
(G_2, G_3, G_1)	x^6
(G_3, G_1, G_2)	x^1
(G_3, G_2, G_1)	x^1

Table 5: The $LRS(\alpha)$ solution set for the WRMS problem in each ordered objective group where $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_6)$ and $\alpha_j = (\alpha^{\inf}, \alpha^{\inf}, \ldots, \alpha^{\inf}) \in \mathbb{R}^8$ for all $j \in I_q$.

3.5.2 Result analysis

We will classify the above eight objectives into three groups. Group 1 concerns people ($G_1 := (f_1, f_2, f_3)$); Group 2 concerns the environment ($G_2 := (f_4, f_5, f_6)$); Group 3 concerns financial matters ($G_3 := (f_7, f_8)$). In each group, we will also consider the priority of the objectives in the group, for example in the group G_1, the regional interest (f_1) is considered as the most important one, the local interest (communities) (f_2) is considered as the second most important one, and the negative effects on the resettlement of people (f_3) is considered as the least important one of the group G_1. In this problem, we show the computation of the objective group (G_1, G_2, G_3). By applying Algorithm 1, we can obtain a lexicographic tolerable robust solution for the WRMS. Table 7 shows the information of function $\hat{c}^{(\cdot)}(x^k)$ of each feasible solution x^k which is obtained from sorting the vector of component function $c^{(i)}(x^k)$ in nonincreasing way over all scenarios s_j, for each $i \in I_p$. Table 8 presents the jth worst performance vector of all feasible solutions and the reference point of this problem is $(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_6^*) \in \mathbb{R}^{8\times 6}$. According to Theorem 3.3.4, we obtain the tolerance threshold $\alpha^{\inf} = 0$. Therefore, the resulting set of lexicographic robust solutions with respect to $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_6)$ is:

$$LRS(\alpha) = \{x^3\},$$

where $\alpha_j = (\alpha^{\inf}, \alpha^{\inf}, \ldots, \alpha^{\inf}) \in \mathbb{R}^8$, for all $j \in I_q$.

3.5.3 Solution sets with different objective priorities

In Table 5, the solution sets of the problem which are corresponding to each group of the ordered objective function are presented. One may observe that, different values for the priorities yield different solution sets.

Remark 3.5.1. (i) An important point to note is that by using the most robust compromise solution concept which was discussed in [22], the solution will be x^5. This means that the output from the lexicographic tolerable robust solution and the output from the most robust compromise solution concept can be quite different. Furthermore, note that the solution set derived from the most robust compromise concept will remain the same, regardless of the permutations of the components of the objective function.

(ii) Another solution concept is the robust efficiency, which was introduced by Ehrgott et al. [8]. Based on the data, which are considered in the WRMS problem, and following the concept of the robust efficiency concept we can see that the solution set is $\{x^1, x^2, x^3, x^4, x^5, x^6\}=\mathcal{S}$. In fact, in [8], each element of the solution set $\mathcal{S}$ can be found by applying the weighted sum scalarization method:

$$(\mathcal{MP}(\mathcal{U}))_{w_l} \qquad \min \max_{j\in\{1,2,\ldots,6\}} \sum_{i=1}^{8} w_l^{(i)} f_i(x^k, s_j) \qquad \text{subject to } x^k \in X, \tag{3.5.2}$$

where $w_l := (w_l^{(1)}, w_l^{(2)}, \ldots, w_l^{(8)}) \in \mathbb{R}^8_{\succ}$. One may use the following weight sets to consider the above single objective optimization problem $(\mathcal{MP}(\mathcal{U}))_{w_l}$:

$w_1 = (691.0782, 458.1161, 165.2403, 249.0968, 91.5001, 221.3457, 484.6561, 455.7014)$,

$w_2 = (831.0456, 43.0179, 48.2109, 258.1919, 29.4128, 526.5054, 264.536, 716.3191)$,

$w_3 = (224.5293, 605.5699, 649.4945, 864.5647, 341.5705, 106.62, 8.2109, 291.8441)$,

$w_4 = (299.4271, 397.7614, 868.3355, 286.74, 781.3634, 129.4872, 9.4937, 891.1759)$,

$w_5 = (952.469, 440.2029, 336.5277, 328.4372, 902.203, 627.7193, 22.8332, 125.6362)$,

$w_6 = (733.3956, 693.8117, 796.0924, 198.2816, 8.0612, 979.1434, 37.3021, 228.8411)$,

and find that the corresponding solutions of weights w_1, w_2, w_3, w_4, w_5, and w_6 are x^1, x^2, x^3, x^4, x^5, and x^6, respectively.

Remark 3.5.1 indicated that the solution sets corresponding to different approaches could provide different solution sets. The following remark shows that more desirable solutions for fitting with the decision maker's preferences can be found by using the method in Theorem 3.3.9 when there is the situation that the group of decision makers do not satisfy with the solution x^3.

Remark 3.5.2. Considering again the data of the WRMS problem and suppose the situation that the solution choice x^3 does not satisfy the group of decision makers. Note that the feasible solution x^3 is considered as the first ranking of solution set. The other rankings of solution set are presented in the Table 9.

We now describe the computations for obtaining the results which are presented in Table 9. The value of the tolerance threshold for each ranking of solution set is computed according to Theorem 3.3.9. The tolerance threshold α_j^2 $(j \in \{1, 2, \ldots, 6\})$ for computing the second ranking is

$$\alpha_j^2 = (1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3) \in \mathbb{R}^8, \text{ for all } j \in \{1, 2, \ldots, 6\}.$$

Subsequently, the solution set that is associated to the tolerance threshold α^2, where $\alpha^2 := (\alpha_1^2, \alpha_2^2, \ldots, \alpha_6^2) \in \mathbb{R}^{8\times 6}$ is $\{x^3, x^6\}$. Thus, as discussed above, we will say that the solution set of the second ranking is $\{x^6\}$.

Next, using again the Theorem 3.3.9, we can found that

$$\alpha_j^3 = (2, 2, 2, 2, 2, 2, 2, 2) \in \mathbb{R}^8, \text{ for all } j \in \{1, 2, \ldots, 6\}.$$

Furthermore, the corresponding solution set of this tolerance threshold is $\{x^3, x^4, x^5, x^6\}$. So, we say that the third ranking of solution set is $\{x^4, x^5\}$. By continuing this idea, the rest of rankings of solution set can be computed and obtained as showing in Table 9.

As we can see in Remark 3.5.2, the corresponding solution set of the third ranking is $\{x^4, x^5\}$, which was shown in Table 9. By applying the idea of sharpening the tolerance threshold in the subsection 3.3.4, the sub-rank of the third ranking can be found.

Remark 3.5.3. Observe that in Remark 3.5.2 by taking a tolerance threshold, $\alpha_j^3 = (2,2,2,2,2,2,2,2)$, for all $j \in I_q$, there are two members in the third ranking of solution set that are x^4 and x^5. By applying the formulation (3.3.8) to refine the tolerance threshold for classifying the sub-rank between x^4 and x^5, we obtain the corresponding sub-rank of the 3^{rd} ranking of the solution set as follows:

$$LRS(\alpha^{3_1}) = \{x^5\}$$

and

$$LRS(\alpha^{3_2}) = \{x^4\}.$$

These imply that the feasible solution x^5 is considered as a robust solution in the first sub-rank and the feasible solution x^4 is considered as a robust solution in the second sub-rank of the third ranking of the solution set, respectively. This means that x^5 is more desirable than x^4.

CHAPTER IV

THE LIGHTLY ROBUST MAX-ORDERING SOLUTIONS

This chapter motivated by the concepts of max-ordering optimality in Definition 2.1.7 and lightly robust optimality in Definition 2.2.7 from Chapter II. Based on these ideas, we propose a new robust concept by relying on both features of those two solution concepts for uncertain multicriteria optimization problems.

To do this, we begin with introducing the notaion $\mathcal{MP}(\hat{s})$ which is used to stand for the nominal problem of uncertain multicriteria optimization problems $\mathcal{MP}(\mathcal{U})$. That is, for an uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$ together with a nominal scenario $\hat{s} \in \mathcal{U}$, the *nominal problem* $\mathcal{MP}(\hat{s})$ is given as a deternimistic multicriteria optimization problem:

$$(\mathcal{MP}(\hat{s})) \qquad \min f(x, \hat{s}) \quad \text{subject to } x \in X \tag{4.0.3}$$

with the objective function $f : \mathbb{R}^n \times \{\hat{s}\} \to \mathbb{R}^p$.

4.1 The lightly robust max-ordering solution concept

To introduce a new solution concept of this chapter, we now give some important notations that are relevant to our solution concept. According to Definition 2.1.7, the set of max-ordering solutions for the nominal problem $\mathcal{MP}(\hat{s})$ can be founded by solving the following optimization problem:

$$\min_{x \in X} \max_{i \in I_p} f_i(x, \hat{s}). \tag{4.1.1}$$

We denote the set of max-ordering solutions to the nominal problem $\mathcal{MP}(\hat{s})$ by $X_{MO}(\hat{s})$.

For any fixed non-negative value ε, we define the robust counterpart of an uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$ as following

$$(\mathcal{LRMOP}(\hat{s},\varepsilon)) \qquad \min_{s\in\mathcal{U}} \max_{i\in I_p} f_i(x,s) \qquad (4.1.2)$$
$$\text{subject to } x \in X_{\mathcal{LRMOP}(\hat{s},\varepsilon)},$$

where $X_{\mathcal{LRMOP}(\hat{s},\varepsilon)} := \{x \in X | \max_{i\in I_p} f_i(x,\hat{s}) \leqslant \max_{i\in I_p} f_i(x',\hat{s}) + \varepsilon\}$, for some $x' \in X_{MO}(\hat{s})$.

We now present the solution concept which is the main aim of this Chapter.

Definition 4.1.1. Given an uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$ with a nominal scenario $\hat{s}$. Let $\varepsilon \geqslant 0$ be given. Then, a feasible solution x^* is called a *lightly robust max-ordering solution* for the problem $\mathcal{MP}(\mathcal{U})$ with respect to the relaxation ε on the nominal scenario $\hat{s}$ if it is an optimal solution for the optimization problem $\mathcal{LRMOP}(\hat{s},\varepsilon)$. The set of all lightly robust max-ordering solutions is denoted by $X^*_{\mathcal{LRMOP}(\hat{s},\varepsilon)}$.

Remark 4.1.2. When considering the relaxation $\varepsilon = 0$, the feasible set of the problem $\mathcal{LRMOP}(\hat{s},0)$ is a set $\{x \in X | \max_{i\in I_p} f_i(x,\hat{s}) \leqslant \max_{i\in I_p} f_i(x',\hat{s})\}$. This means that all elements in the feasible set of the problem $\mathcal{LRMOP}(\hat{s},0)$ being max-ordering solutions for the nominal problem $\mathcal{MP}(\hat{s})$. Ones can see that the corresponding solutions to $\mathcal{LRMOP}(\hat{s},0)$ provide solutions which are prioritized a normal circumstance as the most important situation.

The following remark is the observations on the concept of solution in Definition 4.1.1 in the special cases.

Remark 4.1.3. (i) Note that when $|\mathcal{U}| = 1$, it follows that $\mathcal{MP}(\hat{s}) = \mathcal{MP}$. Then, the solution concept in Definition 4.1.1 is nothing but the concept of max-ordering optimality in Definition 2.1.7 with respect to $\varepsilon = 0$.

(ii) When $p = 1$, the solution concept in Definition 4.1.1 coincides with the concept of lightly optimality in Definition 2.2.7.

(iii) Notice that the optimal value according to the Definition 4.1.1 is greater than or equal to the optimal value of the nominal problem $\mathcal{MP}(\hat{s})$ in the uncertain environments.

In the next section, we present a measurement that shows a visualization of the performance of the proposed solution with respect to the relaxation level.

4.2 The price of robustness

As the solution set according to the problem $\mathcal{LRMOP}(\hat{s}, \varepsilon)$ in Definition 4.1.1 is depending on the relaxation ε, making decision by only relying on this information may not enough for the decision makers. In order to know a trade-off between the robustness of a solution and the quality of a solution with respect to nominal scenario, we provide some additional information which can be used to help decision makers to know how much nominal quality has to be sacrificed to obtain more desirable robust solution. In doing so, we present two measures that can serve as strategies for finding the most desirable solution, which we called the gain in robustness and the price to be paid for robustness. The underlying idea of the first measure approach is used to interpret robustness of the lightly robust max-ordering solution compare with max-ordering solution of nominal problem in the worst case, and the second measure approach is used to explain the price to be paid for robustness in nominal scenario. In lightly robust max-ordering solution method, we calculate the *gain in robustness* as

$$gain(x_\varepsilon^{(\max,light)}, \hat{s}) := \min_{x \in X_{MO}(\hat{s})} \max_{s \in \mathcal{U}} \max_{i \in I_p} f_i(x, s) - \max_{s \in \mathcal{U}} \max_{i \in I_p} f_i(x_\varepsilon^{(\max,light)}, s) \tag{4.2.1}$$

where $x_\varepsilon^{(\max,light)}$ is a lightly robust max-ordering solution with respect to the relaxation ε for the problem $\mathcal{MP}(\mathcal{U})$. Observe that the value of $gain(x_\varepsilon^{(\max,light)}, \hat{s})$ is expressed a visualization of the robustness that $x_\varepsilon^{(\max,light)}$ is better than max-ordering solutions on the worst case scenario. Analogously, we also calculate *the price to be paid for robustness* as

$$price(x_\varepsilon^{(\max,light)}, \hat{s}) \;:=\; \max_{x \in X^*_{\mathcal{LRMOP}(\hat{s},\varepsilon)}} \max_{i \in I_p} f_i(x, \hat{s}) - \max_{i \in I_p} f_i(x', \hat{s}), \tag{4.2.2}$$

for some $x' \in X_{\mathcal{MO}}(\hat{s})$. The value of $price(x_\varepsilon^{(\max,light)}, \hat{s})$ interpret that how much the quality of a lightly robust max-ordering solution $x_\varepsilon^{(\max,light)}$ is losing compare to x' in the nominal problem. The explanation of these measures is that how much nominal quality is lost when we want more robustness of a solution regarding to each relaxation. By considering the ratio of those two measures, the decision makers can make informed decision according to preferences in both aspects.

In practitioner point of view, it is good to choose a solution which works well in both respects is that in the worst case scenario and nominal scenario. Based on the equation (4.2.1) and (4.2.2), we now suggest a method for finding a lightly robust max-ordering solution to the problem. To do so, the steps of finding lightly robust max-ordering solutions are the following:

Algorithm 2: Finding lightly robust max-ordering solutions and its price and gain

Input: Uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$.

Step.1: Choose the nominal scenario $\hat{s}$. Solve the deterministic multicriteria optimization problem $\mathcal{MP}(\hat{s})$ to find a set of max-ordering solutions $X_{\mathcal{MO}}(\hat{s})$.

Step.2: For the relaxation $\varepsilon \geqslant 0$ on the nominal scenario $\hat{s}$, compute the solution set $X^*_{\mathcal{LRMOP}(\hat{s},\varepsilon)}$ of the problem $\mathcal{LRMOP}(\hat{s},\varepsilon)$.

Step.3: Compute the gain in robustness $gain(x_\varepsilon^{(\max,light)}, \hat{s})$ and the price to be paid for robustness $price(x_\varepsilon^{(\max,light)}, \hat{s})$ as the formulations of equations (4.2.1) and (4.2.2), respectively.

Output: Lightly robust max-ordering solutions $x_\varepsilon^{(\max,light)}$ and the gain in robustness $gain(x_\varepsilon^{(\max,light)}, \hat{s})$ and the price to be paid for robustness $price(x_\varepsilon^{(\max,light)}, \hat{s})$.

Remark 4.2.1. (i) In step 1, by applying the algorithm of solving max-ordering optimization problem in [17], we can obtain a max-ordering solution for the deterministic multicriteria optimization problem $\mathcal{LRMOP}(\hat{s},\varepsilon)$.

(ii) Observe that the computation of value ε is finding the infimum of subset of real numbers. Thus, we can apply many elementary existing methods of finding this value.

(iii) Notice that in step 2, solving the problem $\mathcal{LRMOP}(\hat{s},\varepsilon)$ is closely connected with solving a max-ordering scalarization method in [25]. That is, for a given weight vector $\lambda \in \mathbb{R}^p_{\succ}$ and reference point $r \in \mathbb{R}^p$, the corresponding max-ordering optimization problem is

$$(P - max(r,\lambda)) \qquad \min_{x\in X} \max_{s\in\mathcal{U}} \max_{i\in I_p} \lambda_i(f_i(x,s) - r_i),$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ and $r = (r_1, r_2, \ldots, r_p)$. By setting all weighted $\lambda_i = 1$

and the reference point $r_i = 0$, for all $i \in I_p$, we may apply many existing method of solving such problem in [25] and reference therein.

Based on Algorithm 2, the corresponding solution achieved by this method was just solutions for selecting a relaxation ε. In other words, the degree of protection of the obtained solution for uncertainty data is correlated with the choice of such relaxation ε. So, making decision by relying only this information may not enough for decision makers. To find more desirable solutions, the method of calculating the effective relaxation ε will be presented and discussed in the next subsection.

4.2.1 The threshold degradation

This section will be discuss with a relaxation ε for determining a feasible set of the problem $\mathcal{LRMOP}(\hat{s}, \varepsilon)$. It should be noted that the lightly robust max-ordering solution depending on the choice of relaxation ε. If there is a situation that decision makers need more robustness on a solution, the method on how to choose the effective relaxation ε for classifying the level of robustness of the solution set is considered. Here, we consider the idea of classifying the level of robustness of solution set for the proposed solution concept. In doing so, we begin by computing the relaxation for determining the first level of robustness of solution set by taking from the minimal value of the deviation between the maximum value among all objectives of each feasible solution and of optimal solutions in the nominal scenario. After that, the next level of robustness of solution set can be computed by removing all elements that belong to the first level of robustness of solution set from the feasible set. This mentioned idea is presented below in the situation that the feasible solution set X is finite.

Theorem 4.2.2. Let $X \subseteq \mathbb{R}^n$ be a feasible set and function $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^p$. For each $m \in \{2, 3, \dots\}$, let ε^m be defined by

$$\varepsilon^m := \min_{x \in X \setminus X^*_{\mathcal{LRMOP}(\hat{s}, \varepsilon^{m-1})}} \{\max_{i \in I_p} f_i(x, \hat{s}) - \max_{i \in I_p} f_i(x', \hat{s})\}, \tag{4.2.3}$$

where $\varepsilon^1 = 0$. If X is a finite set, then for any $\beta \in [\varepsilon^m, \varepsilon^{m+1})$, we have

$$X^*_{\mathcal{LRMOP}(\hat{s},\varepsilon^m)} = X^*_{\mathcal{LRMOP}(\hat{s},\beta)}. \tag{4.2.4}$$

Proof. We note that the solution set $X^*_{\mathcal{LRMOP}(\hat{s},\varepsilon^m)}$ and $X^*_{\mathcal{LRMOP}(\hat{s},\beta)}$ are results of the same objective function that concern with the feasible sets $X_{\mathcal{LRMOP}(\hat{s},\varepsilon^m)}$ and $X_{\mathcal{LRMOP}(\hat{s},\beta)}$, respectively. Thus, we only need to show that

$$X_{\mathcal{LRMOP}(\hat{s},\varepsilon^m)} = X_{\mathcal{LRMOP}(\hat{s},\beta)}.$$

We firstly show that $X_{\mathcal{LRMOP}(\hat{s},\varepsilon^m)} \subseteq X_{\mathcal{LRMOP}(\hat{s},\beta)}$. Let $x^* \in X_{\mathcal{LRMOP}(\hat{s},\varepsilon^m)}$. This means that

$$\max_{i\in I_p} f_i(x^*, \hat{s}) \leqslant \max_{i\in I_p} f_i(x', \hat{s}) + \varepsilon^m, \text{ for all, } x' \in X_{\mathcal{MO}}(\hat{s}). \tag{4.2.5}$$

Thus, by $\varepsilon^m \leqslant \beta$, it follows that

$$\max_{i\in I_p} f_i(x^*, \hat{s}) \leqslant \max_{i\in I_p} f_i(x', \hat{s}) + \beta, \text{ for all, } x' \in X_{\mathcal{MO}}(\hat{s}). \tag{4.2.6}$$

This implies that $x^* \in X_{\mathcal{LRMOP}(\hat{s},\beta)}$ and so $X_{\mathcal{LRMOP}(\hat{s},\varepsilon^m)} \subseteq X_{\mathcal{LRMOP}(\hat{s},\beta)}$.

Next, we will show $X_{\mathcal{LRMOP}(\hat{s},\beta)} \subseteq X_{\mathcal{LRMOP}(\hat{s},\varepsilon^m)}$. Let $x^* \in X_{\mathcal{LRMOP}(\hat{s},\beta)}$. Suppose on the contrary that $x^* \notin X_{\mathcal{LRMOP}(\hat{s},\varepsilon^m)}$. Thus, by the definition of ε^{m+1}, it would follow that

$$\varepsilon^{m+1} \leqslant \max_{i\in I_p} f_i(x^*, \hat{s}) - \max_{i\in I_p} f_i(x', \hat{s}). \tag{4.2.7}$$

Note that since $x^* \in X_{\mathcal{LRMOP}(\hat{s},\beta)}$, we have

$$\max_{i\in I_p} f_i(x^*, \hat{s}) \leqslant \max_{i\in I_p} f_i(x', \hat{s}) + \beta. \tag{4.2.8}$$

Thus, from the equations (4.2.7) and (4.2.8), we get

$$\varepsilon^{m+1} \leqslant \max_{i\in I_p} f_i(x^*, \hat{s}) - \max_{i\in I_p} f_i(x', \hat{s}) \leqslant \beta. \tag{4.2.9}$$

Which leads to a contradiction with the choice of β. Therefore, we obtain the remaining inclusion and the proof is completed. $\square$

Remark 4.2.3. Notice that for each $m \in \mathbb{N}$, by choice of computing the relaxation ε^m as in formulation (4.2.3), we can see that $\varepsilon^m < \varepsilon^{m+1}$, for each $m \in \mathbb{N}$.

The method for finding the relaxation in Theorem 4.2.2 will leads us to determine the level of robustness of solution set. By applying this method together with the measures of the gain in robustness and the price to be paid for robustness of each solution set, the most desirable solution according to the decision maker's preference can be found.

4.3 Case study: The ambulance location optimization problem

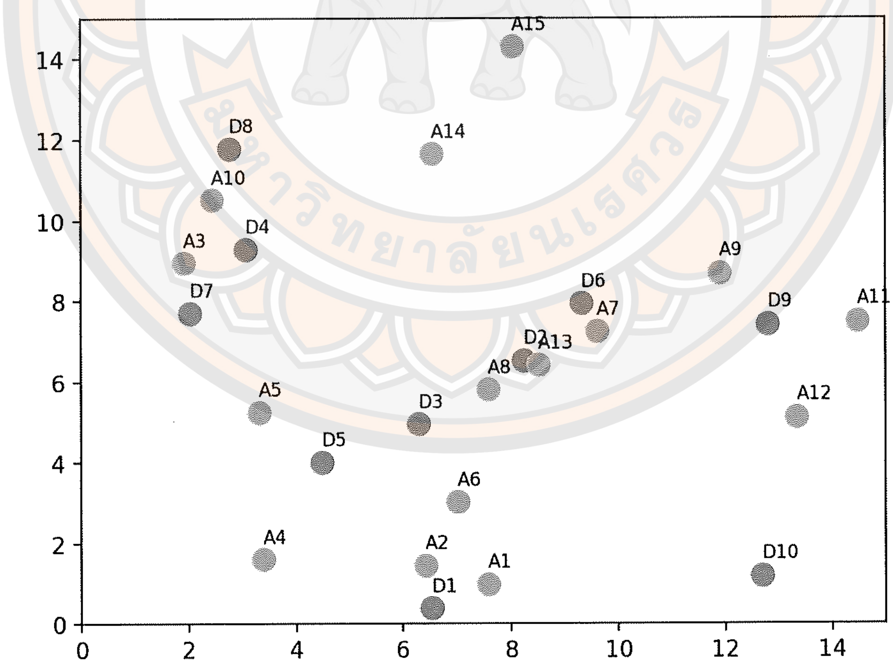

Figure 1: **Ambulance candidate locations and potential demand sites**

In this section, we focus on applying the lightly robust max-ordering solution concept to the ambulance location problem in the emergency medical services system. This demonstration shows that the proposed solution concept can help decision makers to find the optimal location patterns for ambulance car parking stations that could reach to the accident potential demand sites effectively. Based on this approach, a solution provides a minimum value of the maximum distance that covers all demand points. Moreover, by considering the measures of the gain in robustness and price to be paid for robustness, the decision makers can see how much they have to scarify on the nominal quality for obtaining a robustness on lightly robust max-ordering solutions in each level of robustness of solution set.

Suppose that we want to find the suitable stand-by ambulance cars parking patterns for placing 5 ambulances among all 15 possible candidate locations such that the longest distance covering over all 10 demand sites and its closest ambulance is minimal. In Figure 1, the orange one and the blue one are used to indicate the candidate ambulance locations and the potential demand sites in this emergency medical services system, respectively. Here are the notations which will be used throughout this problem.

Notations:

- Let I_{10} be the index set of emergency demand sites
- Let J_{15} be the index set of ambulance candidate locations
- Let H_5 be the index set of the considered ambulances
- D_i represents the emergency demand site i, where $i \in I_{10}$ (the blue one)
- d_i represents the weight of emergency demand site D_i, where $i \in I_{10}$ (the details on each value of d_i can be found on Table 10)
- A_j represents the ambulance candidate location j, where $j \in I_{15}$ (the orange one)
- $a_k := \{a_k^1, a_k^2, a_k^3, a_k^4, a_k^5\}$ denoting the k^{th} location pattern for the considered 5 ambulances.

Looking at the above problem setting of emergency demand sites, we then have the objective function is $f := (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10})$. As the propose of this problem is to find the most effective location patterns from 15 potential candidate locations for placing 5 ambulances, the possible alternative candidate location patterns of this problem are computed from the following formula:

$$\binom{15}{5} = \frac{15!}{(5!)(15-5)!} = 3,003.$$

Here, these patterns are feasible solutions and so the feasible set is $X := \{a_k | k \in E_{3,003}\} \subset \mathbb{R}^5$, where $E_{3,003}$ is the index set of indice k of each possible alternative candidate location pattern.

In order to achieve a reliable ambulance location pattern for rescue operations in the emergency medical services system, the problems of where to locate the ambulance facilities have become the focus of attention. Here we consider the problem of locating the ambulance where the situation of unavailability of the ambulance may be occurred. This study, we assume that all ambulances are same conditions. We consider all the possible events with ambulances simultaneously unavailable. So, all possible events of the considered problem are:

Possible events:

- There is no unavailable ambulance ($\mathcal{U}_0$)
- There is one ambulance unavailability ($\mathcal{U}_1$)
- There are two ambulances simultaneously unavailable ($\mathcal{U}_2$)
- There are three ambulances simultaneously unavailable ($\mathcal{U}_3$)
- There are four ambulances simultaneously unavailable ($\mathcal{U}_4$)

Since there are 5 ambulances to allocate in this system, for each $P \in \{0, 1, 2, 3, 4\}$ a set $\mathcal{U}_P$ of each event is composed of sub-events itself. Here, a sub-event in the set $\mathcal{U}_P$ is considered as scenario. According to the above possible events in this problem, for each candidate location pattern $a_k \in X$ and $P \in \{1, 2, 3, 4\}$, the number of scenarios in each $\mathcal{U}_P^{a_k}$ can be computed by the following formula:

$$|\mathcal{U}_P^{a_k}| = \binom{5}{P} = \frac{5!}{(P!)(5-P)!}, \tag{4.3.1}$$

where the notation P in the formulation (4.3.1) is denoted to the number of ambulances which are simultaneously unavailable. Hence, for each $a_k \in X$, the possible scenarios according to the mentioned five events are:

$$\begin{aligned} |\mathcal{U}_0| + \left[|\mathcal{U}_1^{a_k}| + |\mathcal{U}_2^{a_k}| + |\mathcal{U}_3^{a_k}| + |\mathcal{U}_4^{a_k}|\right] &= 1 + \left[\binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4}\right] \\ &= 31. \end{aligned} \tag{4.3.2}$$

To present more clearly, we denote each scenario of ambulance's unavailability in this system with respect to each candidate location pattern a_k by the following notations:

- $\mathcal{U}_0 = \{s_{\{0\}}\}$
- $\mathcal{U}_1^{a_k} = \{s^k_{\{1\}}, s^k_{\{2\}}, s^k_{\{3\}}, s^k_{\{4\}}, s^k_{\{5\}}\}$
- $\mathcal{U}_2^{a_k} = \{s^k_{\{1,2\}}, s^k_{\{1,3\}}, s^k_{\{1,4\}}, s^k_{\{1,5\}}, s^k_{\{2,3\}}, s^k_{\{2,4\}}, s^k_{\{2,5\}}, s^k_{\{3,4\}}, s^k_{\{3,5\}}, s^k_{\{4,5\}}\}$
- $\mathcal{U}_3^{a_k} = \{s^k_{\{1,2,3\}}, s^k_{\{1,2,4\}}, s^k_{\{1,2,5\}}, s^k_{\{1,3,4\}}, s^k_{\{1,3,5\}}, s^k_{\{1,4,5\}}, s^k_{\{2,3,4\}}, s^k_{\{2,3,5\}}, s^k_{\{2,4,5\}}, s^k_{\{3,4,5\}}\}$
- $\mathcal{U}_4^{a_k} = \{s^k_{\{1,2,3,4\}}, s^k_{\{1,2,3,5\}}, s^k_{\{1,3,4,5\}}, s^k_{\{1,2,4,5\}}, s^k_{\{2,3,4,5\}}\}$.

Note that each scenario subscription refers to the unavailable ambulance labels. For example, the notation $s_{\{0\}}$ refers to there is no unavailable ambulance in this system, the notation $s^k_{\{1\}}$ refers to the 1^{st} label of ambulance is unavailable with respect to the location pattern a_k, and the notation $s^k_{\{1,2\}}$ refers to the 1^{st} label and the 2^{nd} label of ambulances are unavailable with respect to the location pattern a_k in this system.

As the possible candidate location patterns in this problem are $3,003$ patterns, the number of all possible scenarios according to the formulation (4.3.2) is

$$|\mathcal{U}_0| + \binom{15}{5}\left[|\mathcal{U}^{a_k}_1| + |\mathcal{U}^{a_k}_2| + \mathcal{U}^{a_k}_3| + |\mathcal{U}^{a_k}_4|\right] = 1 + (3,003 \times 30) = 90,091.$$

For convinence, we donote the set of all possible scenarios for this problem by

$$\mathcal{U} := \mathcal{U}_0 \bigcup \left(\bigcup_{k=1}^{3,003}\left(\bigcup_{i=1}^{4}\mathcal{U}^{a_k}_i\right)\right).$$

Here, the ambulance location problem is formulated as an uncertain multicriteria optimization problem $\mathcal{MP}(\mathcal{U})$, where $\mathcal{MP}(\mathcal{U})$ is given as a family of $\{\mathcal{MP}(s)|s \in \mathcal{U}\}$ of deterministic multicriteria optimization problem as

$$(\mathcal{MP}(s)) \qquad \min f(a_k, s) \qquad \text{subject to } a_k \in X, \tag{4.3.3}$$

and for each $i \in I_{10}$, the component function $f_i : X \times \mathcal{U} \longrightarrow \mathbb{R}$ is defined as

$$f_i(a_k, s_{\{0\}}) = \min_{h\in H_5} d_i\|a^h_k - D_i\|, \tag{4.3.4}$$

and

$$f_i(a_k, s^j_\square) = \begin{cases} \min\limits_{h\in comp(s^j_\square)} d_i\|a^h_k - D_i\|, & \text{if} \quad k = j, \\ 0, & \text{if} \quad k \neq j. \end{cases} \tag{4.3.5}$$

where $comp(s^j_\square) = H_5 \backslash \square$ and $\|\cdot\|$ is a norm on $\mathbb{R}^2$. This means $f_i(a_k, s^j_\square)$ is defined as the shortest distance of ambulance pattern a_k to demand site D_i under scenario $s^j_\square$. We note that the objective function values of the formulations (4.3.4) and (4.3.5) were generated and computed according to the problem setting as Figure 1 and the measurement unit for this example is kilometer.

Here, the robust counterpart $\mathcal{LRMOP}(\hat{s},\varepsilon)$ as in the formulation (4.1.2) of the ambulance location problem (4.3.3) with respect to the relaxation ε is expressed as follows:

$$(\mathcal{LRMOP}(\hat{s},\varepsilon)) \qquad \min \max_{s\in\mathcal{U}} \max_{i\in I_{10}} f_i(a_k, s) \qquad \text{subject to } a_k \in X_{\mathcal{LRMOP}(\hat{s},\varepsilon)}, \tag{4.3.6}$$

where $X_{\mathcal{LRMOP}(\hat{s},\varepsilon)} := \{a_k \in X | \max_{i\in I_{10}} f_i(a_k, \hat{s}) \leqslant \max_{i\in I_{10}} f_i(a'_k, \hat{s}) + \varepsilon\}$ and $\hat{s}$ is the nominal scenario. Note that the notations a'_k and $f_i(a'_k, \hat{s})$ are indicated for the optimal location pattern in the nominal problem and the longest distance covering all demands sites in nominal scenario.

We will assume that the nominal scenario of this system is $s_{\{0\}}$ because this should be considered as a typical situation (In fact, another scenario can be seen as a nominal scenario depending on which situation we would like to define it as the most important event or the frequent event) and consider the distance in $\mathbb{R}^2$ by computing the Euclidean norm. According to Definition 2.1.7 of max-ordering solutions, we obtain $|X_{MO}(s_{\{0\}})| = 757$ and

$$\max_{i\in I_{10}} f_i(a_k, s_{\{0\}}) = 193.24, \text{ for all } a_k \in X_{MO}(s_{\{0\}}).$$

4.3.1 Solution Discussions

We now describe the computations of the results which are presented in Table 11. As we can see from Table 11, the results on solution sets depend upon a selection of different relaxations ε_m, where $m \in [0.00, 303.25]$. For choice of the relaxation $\varepsilon_0 = 0.00$, there are 757 feasible solutions in a feasible set $X_{\mathcal{LRMOP}(s_{\{0\}},\varepsilon_0)}$ whereas all of them are optimal location patterns for the nominal problem $\mathcal{MP}(s_{\{0\}})$. Considering the unavailability of ambulances in the system, we can now apply the Definition 4.1.1 to identify optimal location patterns to manage this situation. According to the Definition 4.1.1, we obtain that there are 56 optimal location patterns in which the longest travel distances concerning unavailability of ambulances of these optimal location patterns are 496.49 kilometers in the worst case scenario (for more details on elements in the set of 56 location patterns, one can see in the solution set $X^*_{\mathcal{LRMOP}(s_{\{0\}},\varepsilon_0)}$ from Table.12). Note that all solutions in the set $X^*_{\mathcal{LRMOP}(s_{\{0\}},\varepsilon_0)}$ are considered as solutions in the first level of robustness.

By applying the method of computing the relaxation in Theorem 4.2.2, the next level of robustness of solution set is determined by the relaxation $\varepsilon_1 = 11.15$. According to this relaxation, the corresponding optimal location patterns are 56 patterns and the longest travel distance of these location patterns are 496.49 kilometers in the worst case scenario, while the longest travel distances of these optimal location patterns are not more than 204.39 kilometers in nominal situation under ε_1 relaxation. Here, the solution set corresponding to $\varepsilon_1 = 11.15$ is considered as the second level of robustness. We note that the solution set for the second level of robustness is more robust than the solution set for the first level of robustness.

Subsequently, by applying again Theorem 4.2.2, the solution set of next level of robustness is associated to the relaxation value $\varepsilon_2 = 32.30$. According to this relaxation ε_2, the number of optimal location patterns and the longest distance of these location patterns are providing the same results as the previous level of robustness of solution set, that are, 56 optimal location patterns and 496.49 kilometers for the longest travel distances in the worst case scenario. But the longest travel distances of these optimal location patterns are not more than 225.54 kilometers in nominal situation under ε_2 relaxation. We say that the solution set according to the relaxation value $\varepsilon_2 = 32.30$ is the third level of robustness.

Following the above idea, we get the relaxation to determine the fourth level of robustness of the solution is $\varepsilon_3 = 49.23$. Here, the corresponding optimal location patterns of this level of robustness are 20 patterns and the longest travel distances of these optimal location patterns are 471.60 kilometers in the worst case scenario, while the longest travel distances of these optimal location patterns are not more than 242.47 kilometers in nominal situation. We observe that in the situation of there are unavailable ambulances take place in the system, the solutions of the fourth level of robustness provide better results on the longest travel distances than the solutions of the third level of robustness in the worst case scenario.

By continuing this idea, the rest of level of robustness of solution set can be obtained by applying the method on the computing the relaxation in Theorem 4.2.2 as showing in Table 11.

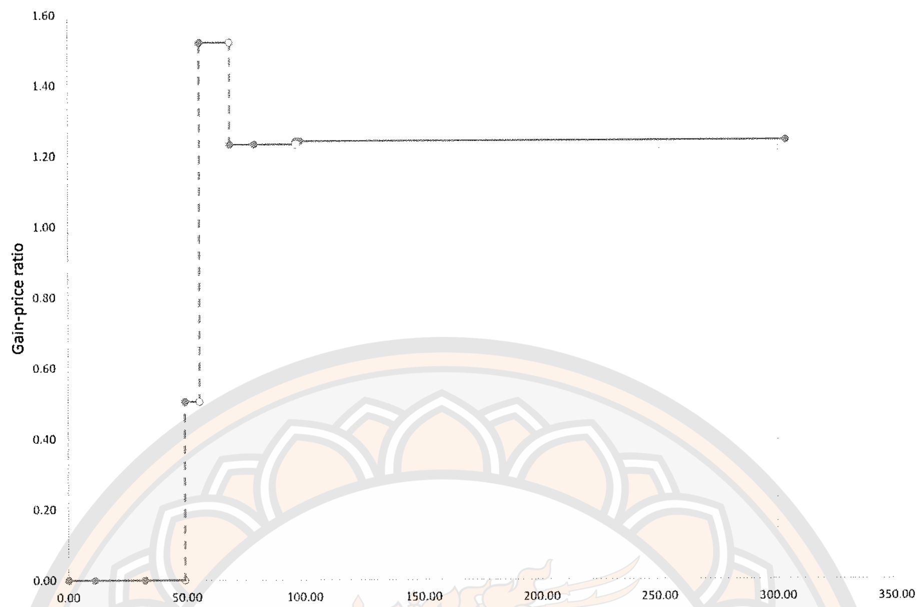

Figure 2: **The ratios $\left(\frac{G}{P}\right)$ of the gain in robustness and the price to be paid for robustness corresponding to different relaxations** $\varepsilon_m \in [00.00, 303.25]$.

Based on the above discussion and the information in Table 11, the question could be arised to decision makers is which relaxation should be chosen. A direction that can be used for obtaining the answer is considering the trade-off between the gain in robustness and the price to be paid for robustness. Here, the ratios which are indicated in Figure 2 can be a useful tool to consider a trade-off in each level of robustness of solution set. Rationally speaking on the ratio of the gain in robustness and the price to be paid for robustness means that the benefits in robustness of solutions which we get, while the nominal quality of solutions we are losing.

From Figure 2, we see that the highest ratio value of trade-off is 1.52 which is obtained from solutions in the fifth level of robustness of the solution set $X^*_{\mathcal{LRMOP}(s_{\{0\}},\varepsilon_4)}$, where $\varepsilon_4 = 55.43$. This means that solutions in the fifth level of robustness can be considered as the desirable than solutions in another level of robustness.

Remark 4.3.1. (i) An important point to note is that if we choose the optimal location pattern relying on just data on the nominal problem $\mathcal{MP}(s_{\{0\}})$ and ignored the uncertainty of unavailable ambulances, it is possible that the network components of location pattern could lose functions when a disaster or crisis occurs in practice. In fact, for example by choice of location pattern $\{A1, A2, A3, A4, A12\}$ which is an optimal solution in the nominal problem (there is neither disaster nor crisis), the longest distance covering all demand sites with respect to this location pattern is 193.24 kilometers. However, if there is the unavailability of ambulances once a vehicle is dispatched to a call, then the longest distance covering all demand sites with respect to this location pattern $\{A1, A2, A3, A4, A12\}$ in the worst case scenario become 644.92 kilometers. Note that this number of the longest distance covering all demand sites by the location pattern $\{A1, A2, A3, A4, A12\}$ is worse than all optimal location patterns which are computed by the concept of lightly robust max-ordering solution in the worst case scenario (more information see in Table 11). This means that the benefits of a solution obtained by our proposed solution concept are ensuring a high performance in serving the longest distance covering all demand sites in the uncertain environments.

(ii) In the general setting on n candidate locations to locate r ambulances, we can calculate all possible scenarios of simultaneously unavailable ambulances by the formula:

$$1 + \binom{n}{r} \sum_{i=1}^{r-1} \binom{r}{i}$$

CHAPTER V

CONCLUSION

This chapter is all the results of this thesis including lemmas and theorems. We conclude again that what we get from the results.

5.1 The lexicographic tolerable robustness concept

This research has extended the concept of lexicographic α-robustness proposed by Kalaï et al. [16] from its original use for uncertain single objective optimization problems to new uses for uncertain multicriteria optimization problems. This new concept of lexicographic robust solution works in situations of uncertainty in which the uncertainty is modelled on a discrete set of scenarios. This new approach is introduced to overcome drawbacks of the minmax robustness approach in the sense of limiting the degree of conservatism of the minmax robustness approach by introducing a tolerance threshold $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q)$. Accordingly, the resulting solution set is obtained from the proposed approach can be guaranteed the immunization of the solution when the decision-making facing of uncertainty and also each performance vector is close to the reference point within the acceptable tolerance threshold. After introducing the fundamentals of this new concept, properties of the solution set and also an algorithm for finding this new kind of solution were presented. The new concept will then be demonstrated on a problem of multicriteria optimization of water resources planning with an uncertainty situation. This water resources planning problem has been selected as an example because the problem's structure is such that each of the multiple objectives carry a different priority. By supposing that some control conditions hold, three Propositions, five Theorems, and two Properties were presented.

Property 5.1.1. **[Dominance]** Let $x \in LRS(\alpha)$. If $y \in X$ satisfies

$$c^{(i)}(y) \lessapprox c^{(i)}(x), \text{ for all } i \in I_p, \tag{5.1.1}$$

then $y \in LRS(\alpha)$.

Property 5.1.2. **[Monotonicity]** The set $LRS(\alpha)$ is monotonic in the tolerance threshold set. That is, for $\alpha := (\alpha_1, \ldots, \alpha_q), \beta := (\beta_1, \ldots, \beta_q) \in \mathbb{R}^{p\times q}$ such that $\alpha_j \lessapprox \beta_j$, for all $j \in I_q$, we have

$$LRS(\alpha) \subseteq LRS(\beta).$$

Proposition 5.1.3. **[Non Preference]** Let $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q) \in [0, \infty)^{p\times q}$. If $x \in LRS(\alpha)$ and $z \notin LRS(\alpha)$, then for each $j \in I_q$, we have

$$worst_j(f(z,\mathcal{U})) \nleqq^{\alpha_j}_{\mathbb{R}^p_{\geqslant}} worst_j(f(x,\mathcal{U})). \tag{5.1.2}$$

Proposition 5.1.4. **[Stability]** For any $x, x' \in LRS(\alpha)$ where $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q) \in [0, \infty)^{p\times q}$, we have

$$worst_j(f(x',\mathcal{U})) \nleqq^{\alpha_j}_{\mathbb{R}^p_{\geqslant}} worst_j(f(x,\mathcal{U}))$$

and

$$worst_j(f(x,\mathcal{U})) \nleqq^{\alpha_j}_{\mathbb{R}^p_{\geqslant}} worst_j(f(x',\mathcal{U}))$$

for all $j \in I_q$.

Proposition 5.1.5. Let X be a feasible set and $\mathcal{MP}(\mathcal{U})$ an uncertain multicriteria optimization problem with the corresponding reference point $\left(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*\right) =: \hat{c}^* \in \mathbb{R}^{p\times q}$. Let $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q) \in \mathbb{R}^{p\times q}$ where $\alpha_j = (\alpha^{\inf}, \alpha^{\inf}, \ldots, \alpha^{\inf}) \in \mathbb{R}^p$, for all $j \in I_q$, such that

$$\alpha^{\inf} := \inf_{x\in X} \max\left(\Delta_x\right), \tag{5.1.3}$$

and $\Delta_x = \begin{bmatrix} worst_1(f(x,\mathcal{U})) - \hat{c}_1^* \\ worst_2(f(x,\mathcal{U})) - \hat{c}_2^* \\ \vdots \\ worst_q(f(x,\mathcal{U})) - \hat{c}_q^* \end{bmatrix} \in \mathbb{R}^{pq}.$

Then, for each $\varepsilon > 0$, we have

(i) $LRS(\alpha + \varepsilon) \neq \emptyset$, and

(ii) $LRS(\alpha - \varepsilon) = \emptyset$.

Theorem 5.1.6. (Nonemptyness) Let $\mathcal{MP}(\mathcal{U})$ be an uncertain multicriteria optimization problem with the corresponding reference point $(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*) := \hat{c}^* \in \mathbb{R}^{p\times q}$, and $f_i(\cdot, s_1), f_i(\cdot, s_2), \ldots, f_i(\cdot, s_q)$ be continuous functions, for all $i \in I_p$. Let $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_q) \in \mathbb{R}^{p\times q}$ where $\alpha_j = (\alpha^{\inf}, \alpha^{\inf}, \ldots, \alpha^{\inf}) \in \mathbb{R}^p$, for all $j \in I_q$, such that a threshold value $\alpha^{\inf}$ is defined as (5.1.3). If X is a compact set then $LRS(\alpha)$ is nonempty.

Theorem 5.1.7. Let $\mathcal{MP}(\mathcal{U})$ be an uncertain multicriteria optimization problem with the corresponding reference point $(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*) := \hat{c}^* \in \mathbb{R}^{p\times q}$, where $\hat{c}_j^* := \left(\hat{c}_j^{*(1)}, \hat{c}_j^{*(2)}, \ldots, \hat{c}_j^{*(p)}\right) \in \mathbb{R}^p$, for all $j \in I_q$. Let $\alpha := (\alpha_1, \ldots, \alpha_q) \in [0,\infty)^{p\times q}$ be such that $\alpha_j := \left(\alpha_j^{(1)}, \alpha_j^{(2)}, \ldots, \alpha_j^{(p)}\right) \in \mathbb{R}^p$ for all $j \in I_q$. Then, we have

$$\bigcap_{(i,j)\in I_p\times I_q} L_{(i,j)} \subseteq LRS(\alpha),$$

where $L_{(i,j)} = \left\{x \in X | \hat{c}_j^{(i)}(x) \leqslant \hat{c}_j^{*(i)} + \alpha_j^{(i)}\right\}$ for all $i \in I_p$ and $j \in I_q$.

Theorem 5.1.8. Let X be a finite set and $f_i(\cdot, s_1), f_i(\cdot, s_2), \ldots, f_i(\cdot, s_q)$ be continuous functions for all $i \in I_p$. For each $m \in \{2, 3, \ldots, q\}$, let α^m defined by

$$\alpha^m := \min_{x\in X\setminus LRS((\alpha^{m-1},\ldots,\alpha^{m-1}),\ldots,(\alpha^{m-1},\ldots,\alpha^{m-1}))} \max(\Delta_x), \tag{5.1.4}$$

where $\alpha^1 := \min_{x\in X} \max(\Delta_x)$. Then, for any $\beta \in [\alpha^m, \alpha^{m+1})$, we have

$$LRS((\beta,\ldots,\beta),\ldots,(\beta,\ldots,\beta)) = LRS((\alpha^m,\ldots,\alpha^m),\ldots,(\alpha^m,\ldots,\alpha^m)).$$

Theorem 5.1.9. Let $\mathcal{MP}(\mathcal{U})$ be an uncertain multicriteria optimization problem with the reference point $(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*) =: \hat{c}^* \in \mathbb{R}^{p\times q}$ and 0 the zero vector in $\mathbb{R}^p$. If $\hat{x} \in A_1^0$ and A_1^0 is defined as in Definition 3.1.2, then $\hat{x}$ is a set-based robust weakly efficient solution for the problem $\mathcal{MP}(\mathcal{U})$.

Theorem 5.1.10. Let $\mathcal{MP}(\mathcal{U})$ be an uncertain multicriteria optimization problem together with reference point $(\hat{c}_1^*, \hat{c}_2^*, \ldots, \hat{c}_q^*) =: \hat{c}^* \in \mathbb{R}^{p \times q}$. Let $\alpha_1 = (\alpha, \alpha, \ldots, \alpha) \in \mathbb{R}^p_{\gtrapprox}$. If $A_1^{\alpha_1} = \{\hat{x}\}$ and $A_1^{\alpha_1}$ is defined as in Definition 3.1.2, then $\hat{x}$ is a set-based minmax robust strictly efficient solution for the problem $\mathcal{MP}(\mathcal{U})$.

5.2 Lightly robust max-ordering solutions

For this section, the new notion of robustness concept called the lightly robust max-ordering solution for uncertain multicriteria optimization problems was proposed. This new robust solution concept is appropriate for decision makers who are interesting in a balancing of the robustness and the nominal quality of robust solutions. The balancing of those two qualities of the proposed solution is interpreted as the measures which are in the formulations of the gain in robustness and the price to be paid for robustness. By introducing these two measures, the decision makers can choose the most desirable robust solution which is satisfied with their own satisfactory. By supposing that some control conditions hold, one Theorem was presented.

Theorem 5.2.1. Let $X \subseteq \mathbb{R}^n$ be a feasible set and a function $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^p$. For each $m \in \{2, 3, \dots\}$, let ε^m defined by

$$\varepsilon^m := \min_{x \in X \setminus X^*_{\mathcal{LRMOP}(\hat{s}, \varepsilon^{m-1})}} \{\max_{i \in I_p} f_i(x, \hat{s}) - \max_{i \in I_p} f_i(x', \hat{s})\}, \tag{5.2.1}$$

where $\varepsilon^1 = 0$. If X is a finite set, then for any $\beta \in [\varepsilon^m, \varepsilon^{m+1})$, we have

$$X^*_{\mathcal{LRMOP}(\hat{s}, \varepsilon^m)} = X^*_{\mathcal{LRMOP}(\hat{s}, \beta)}. \tag{5.2.2}$$

We note that Theorem 5.2.1 provided the method of computing the relaxation which can be used to classify the level of robustness of set of the proposed solution concept. By applying this method together with the measures of the gain in robustness and the price to be paid for robustness in Section 4.2, the most desirable solution with a good trade-off between a performance of a solution in the uncertainty environment and nominal situation can be obtained.

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Table 6: The objective function f of each feasible solution x^k under all scenarios s_j for the WRMS problem.

	$f(\cdot, s_1)$	$f(\cdot, s_2)$	$f(\cdot, s_3)$	$f(\cdot, s_4)$	$f(\cdot, s_5)$	$f(\cdot, s_6)$
$f(x^1, \cdot)$	[-0.70 -1.40 -1.50 -3.60 -5.50 -3.00 369.12 19.30]	[-0.80 -1.00 -0.90 -2.40 -6.50 -2.40 461.40 28.95]	[-1.00 -2.00 -3.00 -2.00 -5.00 -3.00 307.60 19.30]	[-1.40 -2.60 -3.00 -2.40 -1.50 -2.10 307.60 19.30]	[-0.70 -1.40 -1.50 -2.20 -9.00 -3.00 369.12 19.30]	[-0.70 -1.40 -1.50 -3.60 -5.50 -3.00 307.60 23.16]
$f(x^2, \cdot)$	[-1.40 -1.40 -2.00 -3.60 -5.50 -3.00 376.20 17.60]	[-1.60 -1.00 -1.20 -2.40 -6.50 -2.40 470.25 26.40]	[-2.00 -2.00 -4.00 -2.00 -5.00 -3.00 313.50 17.60]	[-2.80 -2.60 -4.00 -2.40 -1.50 -2.10 313.50 17.60]	[-1.40 -1.40 -2.00 -2.20 -9.00 -3.00 376.20 17.60]	[-1.40 -1.40 -2.00 -3.60 -5.50 -3.00 313.50 21.12]
$f(x^3, \cdot)$	[-3.50 -2.80 -2.50 -9.00 -3.30 -4.00 475.08 14.50]	[-4.00 -2.00 -1.50 -6.00 -3.90 -3.20 593.85 21.75]	[-5.00 -4.00 -5.00 -5.00 -3.00 -4.00 395.90 14.50]	[-7.00 -5.20 -5.00 -6.00 -0.90 -2.80 395.90 14.50]	[-3.50 -2.80 -2.50 -5.50 -5.40 -4.00 475.08 14.50]	[-3.50 -2.80 -2.50 -9.00 -3.30 -4.00 395.90 17.40]
$f(x^4, \cdot)$	[-3.50 -2.10 -2.50 -7.20 -4.40 -2.00 454.80 13.70]	[-4.00 -1.50 -1.50 -4.80 -5.20 -1.60 568.50 20.55]	[-5.00 -3.00 -5.00 -4.00 -4.00 -2.00 349.00 13.70]	[-7.00 -3.90 -5.00 -4.80 -1.20 -1.40 379.00 13.70]	[-3.50 -2.10 -2.50 -4.40 -7.20 -2.00 454.80 13.70]	[-3.50 -2.10 -2.50 -7.20 -4.40 -2.00 379.00 16.44]
$f(x^5, \cdot)$	[-3.50 -2.80 -2.50 -7.20 -4.40 -2.00 446.16 14.00]	[-4.00 -2.00 -1.50 -4.80 -5.20 -1.60 557.70 21.00]	[-5.00 -2.00 -5.00 -4.00 -4.00 -2.00 371.80 14.00]	[-7.00 -5.20 -5.00 -4.80 -1.20 -1.40 371.80 14.00]	[-3.50 -2.80 -2.50 -4.40 -7.20 -2.00 446.16 14.00]	[-3.50 -2.80 -2.50 -7.20 -4.40 -2.00 371.80 16.80]
$f(x^6, \cdot)$	[-3.50 -2.10 -2.50 -9.00 -3.30 -5.00 471.72 14.70]	[-4.00 -1.50 -1.50 -6.00 -3.90 -4.00 589.65 22.05]	[-5.00 -1.50 -5.00 -5.00 -3.00 -5.00 393.10 14.70]	[-7.00 -3.90 -5.00 -6.00 -0.90 -3.50 393.10 14.70]	[-3.50 -2.10 -2.50 -5.50 -5.40 -5.00 471.72 14.70]	[-3.50 -2.10 -2.50 -9.00 -3.30 -5.00 393.10 17.64]

Table 7: The sorted vector, $\hat{c}^{(i)}(\cdot)$, of vector $c^{(i)}(\cdot)$ for the WRMS problem.

	$\hat{c}^{(\cdot)}(x^1)$	$\hat{c}^{(\cdot)}(x^2)$	$\hat{c}^{(\cdot)}(x^3)$	$\hat{c}^{(\cdot)}(x^4)$	$\hat{c}^{(\cdot)}(x^5)$	$\hat{c}^{(\cdot)}(x^6)$
$\hat{c}^{(1)}(\cdot)$	$\begin{bmatrix}-0.70\\-0.70\\-0.70\\-0.80\\-1.00\\-1.40\end{bmatrix}$	$\begin{bmatrix}-1.40\\-1.40\\-1.40\\-1.60\\-2.00\\-2.80\end{bmatrix}$	$\begin{bmatrix}-3.50\\-3.50\\-3.50\\-4.00\\-5.00\\-7.00\end{bmatrix}$	$\begin{bmatrix}-3.50\\-3.50\\-3.50\\-4.00\\-5.00\\-7.00\end{bmatrix}$	$\begin{bmatrix}-3.50\\-3.50\\-3.50\\-4.00\\-5.00\\-7.00\end{bmatrix}$	$\begin{bmatrix}-3.50\\-3.50\\-3.50\\-4.00\\-5.00\\-7.00\end{bmatrix}$
$\hat{c}^{(2)}(\cdot)$	$\begin{bmatrix}-1.00\\-1.40\\-1.40\\-1.40\\-2.00\\-2.60\end{bmatrix}$	$\begin{bmatrix}-1.00\\-1.40\\-1.40\\-1.40\\-2.00\\-2.60\end{bmatrix}$	$\begin{bmatrix}-2.00\\-2.80\\-2.80\\-2.80\\-4.00\\-5.20\end{bmatrix}$	$\begin{bmatrix}-1.50\\-2.10\\-2.10\\-2.10\\-3.00\\-3.90\end{bmatrix}$	$\begin{bmatrix}-2.00\\-2.80\\-2.80\\-2.80\\-4.00\\-5.20\end{bmatrix}$	$\begin{bmatrix}-1.50\\-2.10\\-2.10\\-2.10\\-3.00\\-3.90\end{bmatrix}$
$\hat{c}^{(3)}(\cdot)$	$\begin{bmatrix}-0.90\\-1.50\\-1.50\\-1.50\\-3.00\\-3.00\end{bmatrix}$	$\begin{bmatrix}-1.20\\-2.00\\-2.00\\-2.00\\-4.00\\-4.00\end{bmatrix}$	$\begin{bmatrix}-1.50\\-2.50\\-2.50\\-2.50\\-5.00\\-5.00\end{bmatrix}$	$\begin{bmatrix}-1.50\\-2.50\\-2.50\\-2.50\\-5.00\\-5.00\end{bmatrix}$	$\begin{bmatrix}-1.50\\-2.50\\-2.50\\-2.50\\-5.00\\-5.00\end{bmatrix}$	$\begin{bmatrix}-1.50\\-2.50\\-2.50\\-2.50\\-5.00\\-5.00\end{bmatrix}$
$\hat{c}^{(4)}(\cdot)$	$\begin{bmatrix}-2.00\\-2.20\\-2.40\\-2.40\\-3.60\\-3.60\end{bmatrix}$	$\begin{bmatrix}-2.00\\-2.20\\-2.40\\-2.40\\-3.60\\-3.60\end{bmatrix}$	$\begin{bmatrix}-5.00\\-5.50\\-6.00\\-6.00\\-9.00\\-9.00\end{bmatrix}$	$\begin{bmatrix}-4.00\\-4.40\\-4.80\\-4.80\\-7.20\\-7.20\end{bmatrix}$	$\begin{bmatrix}-4.00\\-4.40\\-4.80\\-4.80\\-7.20\\-7.20\end{bmatrix}$	$\begin{bmatrix}-5.00\\-5.50\\-6.00\\-6.00\\-9.00\\-9.00\end{bmatrix}$
$\hat{c}^{(5)}(\cdot)$	$\begin{bmatrix}-1.50\\-5.00\\-5.50\\-5.50\\-6.50\\-9.00\end{bmatrix}$	$\begin{bmatrix}-1.50\\-5.00\\-5.50\\-5.50\\-6.50\\-9.00\end{bmatrix}$	$\begin{bmatrix}-0.90\\-3.00\\-3.30\\-3.30\\-3.90\\-5.40\end{bmatrix}$	$\begin{bmatrix}-1.20\\-4.00\\-4.40\\-4.40\\-5.20\\-7.20\end{bmatrix}$	$\begin{bmatrix}-1.20\\-4.00\\-4.40\\-4.40\\-5.20\\-7.20\end{bmatrix}$	$\begin{bmatrix}-0.90\\-3.00\\-3.30\\-3.30\\-3.90\\-5.40\end{bmatrix}$
$\hat{c}^{(6)}(\cdot)$	$\begin{bmatrix}-2.10\\-2.40\\-3.00\\-3.00\\-3.00\\-3.00\end{bmatrix}$	$\begin{bmatrix}-2.10\\-2.40\\-3.00\\-3.00\\-3.00\\-3.00\end{bmatrix}$	$\begin{bmatrix}-2.80\\-3.20\\-4.00\\-4.00\\-4.00\\-4.00\end{bmatrix}$	$\begin{bmatrix}-1.40\\-1.60\\-2.00\\-2.00\\-2.00\\-2.00\end{bmatrix}$	$\begin{bmatrix}-1.40\\-1.60\\-2.00\\-2.00\\-2.00\\-2.00\end{bmatrix}$	$\begin{bmatrix}-3.50\\-4.00\\-5.00\\-5.00\\-5.00\\-5.00\end{bmatrix}$
$\hat{c}^{(7)}(\cdot)$	$\begin{bmatrix}461.40\\369.12\\369.12\\307.60\\307.60\\307.60\end{bmatrix}$	$\begin{bmatrix}470.25\\376.20\\376.20\\313.50\\313.50\\313.50\end{bmatrix}$	$\begin{bmatrix}593.85\\475.08\\475.08\\395.90\\395.90\\395.90\end{bmatrix}$	$\begin{bmatrix}568.50\\454.80\\454.80\\379.00\\379.00\\379.00\end{bmatrix}$	$\begin{bmatrix}557.70\\446.16\\446.16\\371.80\\371.80\\371.80\end{bmatrix}$	$\begin{bmatrix}589.65\\471.72\\471.72\\393.10\\393.10\\393.10\end{bmatrix}$
$\hat{c}^{(8)}(\cdot)$	$\begin{bmatrix}28.95\\23.16\\19.30\\19.30\\19.30\\19.30\end{bmatrix}$	$\begin{bmatrix}26.40\\21.12\\17.60\\17.60\\17.60\\17.60\end{bmatrix}$	$\begin{bmatrix}21.75\\17.40\\14.50\\14.50\\14.50\\14.50\end{bmatrix}$	$\begin{bmatrix}20.55\\16.44\\13.70\\13.70\\13.70\\13.70\end{bmatrix}$	$\begin{bmatrix}21.00\\16.80\\14.00\\14.00\\14.00\\14.00\end{bmatrix}$	$\begin{bmatrix}22.05\\17.64\\14.70\\14.70\\14.70\\14.70\end{bmatrix}$

Table 8: The jth worst performance vector of each feasible solution x^k and the reference point $\hat{c}_j^*$ for the WRMS problem.

	$worst_1(f(\cdot,\mathcal{U}))$	$worst_2(f(\cdot,\mathcal{U}))$	$worst_3(f(\cdot,\mathcal{U}))$	$worst_4(f(\cdot,\mathcal{U}))$	$worst_5(f(\cdot,\mathcal{U}))$	$worst_6(f(\cdot,\mathcal{U}))$
$worst_{(\cdot)}(f(x^1,\mathcal{U}))$	-0.70	-0.70	-0.70	-0.80	-1.00	-1.40
	-1.00	-1.40	-1.40	-1.40	-2.00	-2.60
	-0.90	-1.50	-1.50	-1.50	-3.00	-3.00
	-2.00	-2.20	-2.40	-2.40	-3.60	-3.60
	-1.50	-5.00	-5.50	-5.50	-6.50	-9.00
	-2.10	-2.40	-3.00	-3.00	-3.00	-3.00
	461.40	369.12	369.12	307.60	307.60	307.60
	28.95	23.16	19.30	19.30	19.30	19.30
$worst_{(\cdot)}(f(x^2,\mathcal{U}))$	-1.40	-1.40	-1.40	-1.60	-2.00	-2.80
	-1.00	-1.40	-1.40	-1.40	-2.00	-2.60
	-1.20	-2.00	-2.00	-2.00	-4.00	-4.00
	-2.20	-2.20	-2.40	-2.40	-3.60	-3.60
	-1.50	-5.00	-5.50	-5.50	-6.50	-9.00
	-2.10	-2.40	-3.00	-3.00	-3.00	-3.00
	470.25	376.20	376.20	313.50	313.50	313.50
	26.40	17.60	17.60	17.60	17.60	17.60
$worst_{(\cdot)}(f(x^3,\mathcal{U}))$	-3.50	-3.50	-3.50	-4.00	-5.00	-7.00
	-2.00	-2.80	-2.80	-2.80	-4.00	-5.20
	-1.50	-2.50	-2.50	-2.50	-5.00	-5.00
	-5.00	-5.50	-6.00	-6.00	-9.00	-9.00
	-0.90	-3.00	-3.30	-3.30	-3.90	-5.40
	-2.80	-3.20	-4.00	-4.00	-4.00	-4.00
	593.85	475.08	475.08	395.90	395.90	395.90
	21.75	17.40	14.50	14.50	14.50	14.50
$worst_{(\cdot)}(f(x^4,\mathcal{U}))$	-3.50	-3.50	-3.50	-4.00	-5.00	-7.00
	-1.50	-2.10	-2.10	-2.10	-3.00	-3.90
	-1.50	-2.50	-2.50	-2.50	-5.00	-5.00
	-4.00	-4.40	-4.80	-4.80	-7.20	-7.20
	-1.20	-4.00	-4.40	-4.40	-5.20	-7.20
	-1.40	-1.60	-2.00	-2.00	-2.00	-2.00
	568.50	454.80	454.80	379.00	379.00	379.00
	20.55	16.44	13.70	13.70	13.70	13.70
$worst_{(\cdot)}(f(x^5,\mathcal{U}))$	-3.50	-3.50	-3.50	-4.00	-5.00	-7.00
	-2.00	-2.80	-2.80	-2.80	-4.00	-5.20
	-1.50	-2.50	-2.50	-2.50	-5.00	-5.00
	-4.00	-4.40	-4.80	-4.80	-7.20	-7.20
	-1.20	-4.00	-4.40	-4.40	-5.20	-7.20
	-1.40	-1.60	-2.00	-2.00	-2.00	-2.00
	557.70	444.16	446.16	371.80	371.80	371.80
	21.00	16.80	14.00	14.00	14.00	14.00
$worst_{(\cdot)}(f(x^6,\mathcal{U})$	-3.50	-3.50	-3.50	-4.00	-5.00	-7.00
	-1.50	-2.10	-2.10	-2.10	-3.00	-3.90
	-1.50	-2.50	-2.50	-2.50	-5.00	-5.00
	-5.00	-5.50	-6.00	-6.00	-9.00	-9.00
	-0.90	-3.00	-3.30	-3.30	-3.90	-5.40
	-3.50	-4.00	-5.00	-5.00	-5.00	-5.00
	589.65	471.72	471.72	393.10	393.10	393.10
	22.05	17.64	14.70	14.70	14.70	14.70

	$worst_1(f(\cdot,\mathcal{U}))$	$worst_2(f(\cdot,\mathcal{U}))$	$worst_3(f(\cdot,\mathcal{U}))$	$worst_4(f(\cdot,\mathcal{U}))$	$worst_5(f(\cdot,\mathcal{U}))$	$worst_6(f(\cdot,\mathcal{U}))$
	-3.50	-3.50	-3.50	-4.00	-5.00	-7.00
	-2.00	-2.80	-2.80	-2.80	-4.00	-5.20
	-1.50	-2.50	-2.50	-2.50	-5.00	-5.00
$\hat{c}_j^*$	-5.00	-5.50	-6.00	-6.00	-9.00	-9.00
	-0.90	-3.00	-3.30	-3.30	-3.90	-5.40
	-2.80	-3.20	-4.00	-4.00	-4.00	-4.00
	593.85	475.08	475.08	395.90	395.90	395.90
	21.75	17.40	14.50	14.50	14.50	14.50

Table 9: The set $LRS(\alpha^i)$ for the (G_1, G_2, G_3) objective group with respect to different tolerence threshold sets where $\alpha^i := (\alpha_1^i, \alpha_2^i, \ldots, \alpha_6^i)$.

Tolerence Threshold Set	$\boldsymbol{LRS(\alpha^i)}$
$\{\alpha_j^1 = (0,0,0,0,0,0,0,0) \mid \forall j = 1,\ldots,6\}$	$\{x^3\}$
$\{\alpha_j^2 = (1.3,1.3,1.3,1.3,1.3,1.3,1.3,1.3) \mid \forall j = 1,\ldots,6\}$	$\{x^3\} \cup \{x^6\}$
$\{\alpha_j^3 = (2,2,2,2,2,2,2,2) \mid \forall j = 1,\ldots,6\}$	$\{x^3\} \cup \{x^6\} \cup \{x^4, x^5\}$
$\{\alpha_j^4 = (5.4,5.4,5.4,5.4,5.4,5.4,5.4,5.4) \mid \forall j = 1,\ldots,6\}$	$\{x^3\} \cup \{x^6\} \cup \{x^4, x^5\} \cup \{x^2\}$
$\{\alpha_j^5 = (7.2,7.2,7.2,7.2,7.2,7.2,7.2,7.2) \mid \forall j = 1,\ldots,6\}$	$\{x^3\} \cup \{x^6\} \cup \{x^4, x^5\} \cup \{x^2\} \cup \{x^1\}$

Table 10: Weight of Demand site D_i for $i \in I_{10}$ of the ambulance location problem

Demand Sites	**Weight of Demand site D_i**
d_1	27.21040801
d_2	4.10474611
d_3	18.31712008
d_4	42.5425252
d_5	20.31375215
d_6	1.36011829
d_7	12.35886195
d_8	3.35721854
d_9	49.69260057
d_{10}	48.52901567

Table 11: Computational experiments of the problem $\mathcal{LRMOP}(s_{\{0\}}, \varepsilon_m)$ where $\varepsilon_m \in [0.00, 303.25]$

Relaxation	**Information**			**Trade-off**		
	$\mid X_{\mathcal{LRMOP}(s_{\{0\}},\varepsilon_m)} \mid$	$\mid X^*_{\mathcal{LRMOP}(s_{\{0\}},\varepsilon_m)} \mid$	Opt. values	Gain (G)	Price (P)	Ratio $\left(\frac{G}{P}\right)$
$\varepsilon_0 = 00.00$	757	56	496.49	0	0	0
$\varepsilon_1 = 11.15$	776	56	496.49	0	0	0
$\varepsilon_2 = 32.30$	791	56	496.49	0	0	0
$\varepsilon_3 = 49.21$	875	20	471.60	24.90	49.21	0.5
$\varepsilon_4 = 55.43$	1395	4	412.07	84.42	55.43	1.52
$\varepsilon_5 = 68.44$	1486	5	412.07	84.42	68.44	1.23
$\varepsilon_6 = 78.74$	1542	5	412.07	84.42	68.44	1.23
$\varepsilon_7 = 96.45$	1987	1	376.69	119.80	96.45	1.24
$\varepsilon_8 = 98.10$	2052	1	376.69	119.80	96.45	1.24
$\varepsilon_9 = 303.25$	3003	1	376.69	119.80	96.45	1.24

Table 12: Optimal location patterns with different relaxations

Thresholds	$X^*_{\mathcal{LRMOP}(s_{\{0\}},\varepsilon_m)}$
$\varepsilon_0, \varepsilon_1, \varepsilon_2 \in [00.00, 32.30]$	$\{A1, A2, A5, A6, A12\}$
	$\{A1, A2, A5, A7, A12\}$
	$\{A1, A2, A5, A8, A12\}$
	$\{A1, A2, A5, A9, A12\}$
	$\{A1, A2, A5, A11, A12\}$
	$\{A1, A2, A5, A12, A13\}$
	$\{A1, A5, A6, A7, A12\}$
	$\{A1, A5, A6, A8, A12\}$
	$\{A1, A5, A6, A9, A12\}$
	$\{A1, A5, A6, A11, A12\}$
	$\{A1, A5, A6, A12, A13\}$
	$\{A1, A5, A7, A8, A12\}$
	$\{A1, A5, A7, A9, A12\}$
	$\{A1, A5, A7, A11, A12\}$
	$\{A1, A5, A7, A12, A13\}$
	$\{A1, A5, A8, A9, A12\}$
	$\{A1, A5, A8, A11, A12\}$
	$\{A1, A5, A8, A12, A13\}$
	$\{A1, A5, A9, A11, A12\}$
	$\{A1, A5, A9, A12, A13\}$
	$\{A1, A5, A11, A12, A13\}$
	$\{A2, A5, A6, A7, A12\}$
	$\{A2, A5, A6, A8, A12\}$

Thresholds	$X^*_{\mathcal{LRMOP}(s_{\{0\}},\varepsilon_m)}$
$\varepsilon_0, \varepsilon_1, \varepsilon_2 \in [00.00, 32.30]$	$\{A2, A5, A6, A9, A12\}$
	$\{A2, A5, A6, A11, A12\}$
	$\{A2, A5, A6, A12, A13\}$
	$\{A2, A5, A7, A8, A12\}$
	$\{A2, A5, A7, A9, A12\}$
	$\{A2, A5, A7, A11, A12\}$
	$\{A2, A5, A7, A12, A13\}$
	$\{A2, A5, A8, A9, A12\}$
	$\{A2, A5, A8, A11, A12\}$
	$(\{A2, A5, A8, A12, A13\}$
	$\{A2, A5, A9, A11, A12\}$
	$\{A2, A5, A9, A12, A13\}$
	$\{A2, A5, A11, A12, A13\}$
	$\{A5, A6, A7, A8, A12\}$
	$\{A5, A6, A7, A9, A12\}$
	$\{A5, A6, A7, A11, A12\}$
	$\{A5, A6, A7, A12, A13\}$
	$\{A5, A6, A8, A9, A12\}$
	$\{A5, A6, A8, A11, A12\}$
	$\{A5, A6, A8, A12, A13\}$
	$\{A5, A6, A9, A11, A12\}$
	$\{A5, A6, A9, A12, A13\}$
	$\{A5, A6, A11, A12, A13\}$
	$\{A5, A7, A8, A9, A12\}$

Thresholds	$X^*_{\mathcal{LRMOP}(s_{\{0\}},\varepsilon_m)}$
$\varepsilon_0, \varepsilon_1, \varepsilon_2 \in [00.00, 32.30]$	$\{A5, A7, A8, A11, A12\}$
	$\{A5, A7, A8, A12, A13\}$
	$\{A5, A7, A9, A11, A12\}$
	$\{A5, A7, A9, A12, A13\}$
	$\{A5, A7, A11, A12, A13\}$
	$\{A5, A8, A9, A11, A12\}$
	$\{A5, A8, A9, A12, A13\}$
	$\{A5, A8, A11, A12, A13\}$
	$\{A5, A9, A11, A12, A13\}$
$\varepsilon_3 = 49.21$	$\{A1, A2, A6, A8, A12\}$
	$\{A1, A2, A7, A8, A12\}$
	$\{A1, A2, A8, A9, A12\}$
	$\{A1, A2, A8, A12, A13\}$
	$\{A1, A6, A7, A8, A12\}$
	$\{A1, A6, A8, A9, A12\}$
	$\{A1, A6, A8, A12, A13\}$
	$\{A1, A7, A8, A9, A12\}$
	$\{A1, A7, A8, A12, A13\}$
	$\{A1, A8, A9, A12, A13\}$
	$\{A2, A6, A7, A8, A12\}$
	$\{A2, A6, A8, A9, A12\}$
	$\{A2, A6, A8, A12, A13\}$
	$\{A2, A7, A8, A9, A12\}$

Thresholds	$X^*_{\mathcal{LRMOP}(s_{\{0\}},\varepsilon_m)}$
$\varepsilon_3 = 49.21$	$\{A2, A7, A8, A12, A13\}$
	$\{A2, A8, A9, A12, A13\}$
	$\{A6, A7, A8, A9, A12\}$
	$\{A6, A7, A8, A12, A13\}$
	$\{A6, A8, A9, A12, A13\}$
	$\{A7, A8, A9, A12, A13\}$
$\varepsilon_4 = 55.43$	$\{A1, A6, A7, A8, A9\}$
	$\{A1, A6, A7, A8, A13\}$
	$\{A1, A6, A8, A9, A13\}$
	$\{A1, A7, A8, A9, A13\}$
$\varepsilon_5, \varepsilon_6 \in [68.44, 78.74]$	$\{A1, A6, A7, A8, A9\}$
	$\{A1, A6, A7, A8, A13\}$
	$\{A1, A6, A7, A9, A13\}$
	$\{A1, A6, A8, A9, A13\}$
	$\{A1, A7, A8, A9, A13\}$
$\varepsilon_7 = 96.45$	$\{A6, A7, A8, A9, A13\}$
$\varepsilon_8 = 98.10$	$\{A6, A7, A8, A9, A13\}$
$\varepsilon_9 = 303.25$	$\{A6, A7, A8, A9, A13\}$