CHAPTER II

PRELIMINARIES

2.1 Permutations

In this section, we introduce some concepts of the permutation group and its properties.

Definition 2.1.1. Let A be a set of elements, a *permutation* of A is a one-to-one (1-1) function (mapping) from A onto A.

If $\{1, 2, \ldots, n\}$ is a set of n elements, then the permutation of this set is written as

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ 1\sigma & 2\sigma & \ldots & n\sigma \end{pmatrix}.$$

The rearrangement of columns in the above symbol for a permutation is immaterial. For example

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

Theorem 2.1.1. [7, pp. 21-22] The set of all permutations on a set A forms a group under the operation of composition, and denoted this group by $\langle S_A, \circ \rangle$.

Definition 2.1.2. If $A = \{1, 2, \ldots, n\}$ then the group of all permutations of A is the *symmetric group of n letters*, and is denoted by " S_n".

Note that S_n has $n!$ elements, where $n! = n(n-1)(n-2)\ldots(3)(2)(1)$.

Example 2.1.1. Find the elements of S_3 and show the group table of S_3.

Solution. S_3 has $3! = 3.2.1 = 6$ elements as follows

$$\sigma_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

The following is the group tables of S_3

$S_3 : \circ$	σ_0	σ_1	σ_2	τ_1	τ_2	τ_3
σ_0	σ_0	σ_1	σ_2	τ_1	τ_2	τ_3
σ_1	σ_1	σ_2	σ_0	τ_2	τ_3	τ_1
σ_2	σ_2	σ_0	σ_1	τ_3	τ_1	τ_2
τ_1	τ_1	τ_3	τ_2	σ_0	σ_2	σ_1
τ_2	τ_2	τ_1	τ_3	σ_1	σ_0	σ_2
τ_3	τ_3	τ_2	τ_1	σ_2	σ_1	σ_0

□

There is another notation for a permutation which is often used.

Definition 2.1.3. A permutation $\sigma \in S_n$ is a *cycle of length r* if there exists distinct integers $i_1, i_2, \ldots, i_r$ such that

$$i_1\sigma = i_2, \quad i_2\sigma = i_3, \quad \ldots, \quad i_{r-1}\sigma = i_r, \quad i_r\sigma = i_1$$

and $m\sigma = m$, if $m \notin \{i_1, i_2, \ldots, i_r\}$. We write such a cycle as $(i_1, i_2, \ldots, i_r)$ or $(i_1 i_2 \ldots i_r)$.

A cycle of length 1 is clearly the identity permutation.

Note that

$$(i_1 i_2 \ldots i_r) = (i_2 \ldots i_r i_1) = (i_3 \ldots i_r i_1 i_2) = \cdots = (i_r i_1 i_2 \ldots i_{r-1}).$$

Example 2.1.2.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (2341),$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 2 & 3 \end{pmatrix} = (15342),$$

$$\text{and} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix} = (231)(4)(5) = (231).$$

□

Example 2.1.3. Let (1456) and (215) be cycles in the group S_6 of all permutations of $\{1, 2, 3, 4, 5, 6\}$

$$\begin{aligned} (1456)(215) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 6 & 5 \end{pmatrix}, \\ (215)(1456) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 5 & 2 & 1 \end{pmatrix}. \end{aligned}$$

Hence $(1456)(215) \neq (215)(1456)$. □

Definition 2.1.4. Two cycles $(i_1 i_2 \ldots i_r)$ and $(j_1 j_2 \ldots j_s)$ are *disjoint* if

$$\{i_1, i_2, \ldots, i_r\} \cap \{j_1, j_2, \ldots, j_s\} = \emptyset.$$

Example 2.1.4. Consider the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 4 & 3 & 1 \end{pmatrix} = (16)(253).$$

Multiplication of two disjoint cycles is clearly commutative, so the order of the factor (16) and (253) is not important. □

Definition 2.1.5. A cycle of length 2 is a *transposition.*

Theorem 2.1.2. [7, p. 27] Every cycle is a product of transpositions.

Example 2.1.5. Show that the expression of a permutation as a product of transpositions is not generally unique

$$\begin{aligned} (1234) &= (12)(13)(14) \\ &= (14)(24)(34) \\ &= (32)(12)(14) \\ &= (13)(24)(34)(12)(24). \end{aligned}$$

□

Theorem 2.1.3. [7, pp. 27-28] If a permutation $\sigma \in S_n$ is a product of k transpositions and also a product of m transpositions then either k and m are both even or both odd.

Definition 2.1.6. A permutation is even or odd according it is a product of an *even* or *odd* number of transpositions.

Example 2.1.6. Examine the following permutation for being even or odd

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 6 & 5 & 1 & 7 & 2 \end{pmatrix}$$

Solution. We first express σ as a product of cycles. So we write

$$\sigma = (145)(2367) = (14)(15)(23)(26)(27)$$

as a product of 5 transpositions. This shows that σ is an odd permutation. $\square$

Theorem 2.1.4. [7, p. 30] Of the $n!$ permutations on n symbols, half are even and half are odd.

Theorem 2.1.5. [7, pp. 30-31] If $n \geq 2$ the collection of all even permutations of finite set of n elements $A_n = \{\sigma \in S_n \mid \sigma \text{ is even}\}$ forms a subgroup of order $n!/2$ of the symmetric group S_n, and it is called the *alternating group* on n letters.

For example,

$$A_3 = \{\iota, (123), (132)\}.$$

$$A_4 = \{\iota, (234), (243), (134), (143), (124), (142), (123), (132), (12)(34), (14)(23), (13)(24)\}.$$

2.2 Permutation Matrices

In this section, we introduce some concepts of the permutation matrices and its properties.

Definition 2.2.1. [8, p. 14] A *permutation matrix* $P \in M_n(\mathbb{R})$ is the identity I with its rows permuted (reordered).

We can also think of a permutation matrix as an identity matrix whose columns have been reordered. Formally:

Definition 2.2.2. [8, p. 14] A matrix $P \in M_n(\mathbb{R})$ is a *permutation matrix* if it contains a single 1 in each column and in each row, and 0 everywhere else.

Example 2.2.1. Consider the effect of multiplying of a 4×4 matrix A by concrete permutation matrix P.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$$

Multiplying by the permutation matrix P on the left, we obtain a new matrix, where the rows of initial the matrix are reordered exactly in the same way as the rows of the identity I are reordered for getting P. Multiplying on the right,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{14} & a_{13} & a_{11} & a_{12} \\ a_{24} & a_{23} & a_{21} & a_{22} \\ a_{34} & a_{33} & a_{31} & a_{32} \\ a_{44} & a_{43} & a_{41} & a_{42} \end{bmatrix}$$

we obtain a new matrix, where the columns of the initial matrix are reordered in the same way as the columns of the identity I are reordered for getting P. □

A matrix P obtained from I by a finite (possibly vacuous) sequence of row swaps is called a permutation matrix. In other words, a permutation matrix is a matrix $P \in M_n(\mathbb{R})$ such that there are row swap matrices $S_1, S_2, \ldots, S_k \in M_n(\mathbb{R})$ for which $P = S_1S_2 \dots S_k$. (Recall that a row swap matrix is by definition an elementary matrix obtained by interchanging two rows of I.) Clearly, I is a permutation matrix, and any product of permutation matrices is also a permutation matrix. It remains to see that the inverse of a permutation matrix is also a permutation matrix. Let $P = S_1S_2 \dots S_k$ be a permutation matrix. Then $P^{-1} = S_k^{-1} \dots S_1^{-1}$. But every row swap S has the property that $S = S^{-1}$, so P^{-1} is indeed a permutation matrix, namely $P^{-1} = S_k \dots S_1$.

Definition 2.2.3. [8, p. 16] The set of all $n \times n$ orthogonal matrices is denoted by $O(n;\mathbb{R})$. We call $O(n, \mathbb{R})$ the *orthogonal group*.

Proposition 2.2.1. [8, p. 16] $O(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$ where

$$GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A^{-1} \text{ exists}\}.$$

Let $P(n)$ denote the set of $n \times n$ permutation matrices. One can also describe $P(n)$ as the set of all matrices obtained from I by permuting the rows of I. Thus $P(n)$ is the set of all $n \times n$ matrices whose only entries are 0 or 1 such that every row and every column has exactly one non-zero entry. It follows from elementary combinatorics that $P(n)$ has exactly $n!$ elements. The inverse of a permutation matrix has a beautiful expression, $P^{-1} = P^T$, so $P(n)$ is a subgroup of the orthogonal group $O(n, \mathbb{R})$, also $P(n)$ is a subgroup of the linear group $GL(n, \mathbb{R})$.

Example 2.2.2. From Example 2.1.1, there are one-to-one correspondent between permutation in $\{\sigma_0, \sigma_1, \sigma_2, \tau_1, \tau_2, \tau_3\}$ and some matrices $\{A_0, A_1, A_2, B_1, B_2, B_3\}$ as the following, respectively.

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det A_0 = 1, \qquad A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \det A_1 = 1,$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \det A_2 = 1, \qquad B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \det B_1 = -1$$

$$B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \det B_2 = -1, \quad B_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det B_3 = -1.$$

□

Theorem 2.2.2. [8, p. 13] If $A \in M_n(\mathbb{R})$ is an orthogonal matrix then $\det A = \pm 1$.

Definition 2.2.4. An orthogonal matrix A with $\det A = 1$ is a rotation matrix.

Definition 2.2.5. An orthogonal matrix A with $\det A = -1$ is a reflection matrix.

In Example 2.2.2, A_0, A_1, A_2 are rotation matrices and B_1, B_2, B_3 are reflection matrices.

Definition 2.2.6. Matrices which are $n \times 1$ or $1 \times n$ are especially called *vectors* and are often denoted by a bold letter. Thus

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

is a $n \times 1$ matrix also called a *column vector* while a $1 \times n$ matrix of the form $\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$ is referred to as a *row vector*.

Theorem 2.2.3. [1, pp. 346-348] If $n \times n$ matrix A has one of the following properties, then the following statement are equivalences:

1. The row vectors of A are linearly independent,
2. The column vectors of A are linearly independent,
3. A is invertible.

Theorem 2.2.4. [3, pp.131-132] Equivalent $\mathbb{C}$-matrices have the same rank.