CHAPTER III

MAIN RESULT

3.1 Some Auxiliary Results

In this section, we give some condition for nonsingularity of sum of two permutation matrices.

Lemma 3.1.1. If $\sigma = (rs)$ be a transposition in S_n where $1 \leq r < s \leq n$, and let $A \in M_n(\mathbb{R})$ be the permutation matrix corresponded to σ, then the system of linear equation $(I + A)\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

Proof.

$$A_\sigma = \begin{array}{r} \\ \\ s\text{- Row} \\ \\ \\ r\text{- Row} \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccc} & & s\text{- Column} & & & r\text{- Column} & & & \end{array} \\ \left[\begin{array}{ccccccccc} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 0 & & & 1 & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & 1 & & & 0 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{array}\right] \end{array}. \tag{3.1.1}$$

Let $I \in M_n(\mathbb{R})$ be the identity matrix, we have

$$I + A_\sigma = \begin{array}{r} \\ \\ s\text{- Row} \\ \\ \\ r\text{- Row} \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccc} & & s\text{- Column} & & & r\text{- Column} & & & \end{array} \\ \left[\begin{array}{ccccccccc} 2 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & 1 & & & \\ & & & 2 & & & & & \\ & & & & \ddots & & & & \\ & & 1 & & & 1 & & & \\ & & & & & & 2 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 2 \end{array}\right] \end{array}. \tag{3.1.2}$$

Now consider a matrix equation

$$(I + A_\sigma)\mathbf{x} = \bar{0} \tag{3.1.3}$$

where

$$\mathbf{x} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}, \quad \text{and} \quad \bar{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in M_{n,1}(\mathbb{R})$$

we get the corresponding homogeneous system of linear equations

$$\begin{cases} 2k_1 & & & & = 0, \\ & \ddots & & & \vdots \\ & k_r & + k_s & & = 0, \\ & & \ddots & & \vdots \\ & k_s & + k_r & & = 0, \\ & & & \ddots & \vdots \\ & & & 2k_n & = 0. \end{cases} \tag{3.1.4}$$

It is easy to see that (rank $I + A_\sigma = n - 1 < n$), therefore the homogeneous system of linear equations (3.1.3) or (3.1.4) is consistent and it has infinitely many solutions.

Similarly, if $\sigma \in S_n$ is product of two or more disjoint transpositions $\tau_1, \tau_2, \ldots, \tau_k$, i.e. $\sigma = \tau_1, \tau_2, \ldots, \tau_k$, and $A_\sigma \in M_n(\mathbb{R})$ is the corresponding permutation matrix of σ then the homogeneous systems $(I + A_\sigma)\mathbf{x} = \bar{0}$ is also has infinitely many solutions. □

Now, suppose $\sigma = (j_1 j_2 j_3)$ is a cycle of length 3 where $1 \leq j_1, j_2, j_3 \leq n$ and A is the permutation matrix corresponding to σ, the homogeneous system of linear equations $(I - A_\sigma)\mathbf{x} = \bar{0}$ must has unique solution that is

$$k_1 = k_2 = \cdots = k_n = 0.$$

For example, let $\sigma = (123) \in S_4$ and let $A \in M_4$ be the corresponding permutation matrix. We have

$$A_\sigma = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad I + A_\sigma = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

We must show that $\det(I + A_\sigma) \neq 0$ equivalently the row vectors of $I + A_\sigma$, that are $\mathrm{e}_1 + \mathrm{e}_3, \mathrm{e}_1 + \mathrm{e}_2, \mathrm{e}_2 + \mathrm{e}_3, 2\mathrm{e}_4$ are linearly independent, where $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4\}$ is the standard basis for $\mathbb{R}^4$.

Let $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$c_1(\mathrm{e}_1 + \mathrm{e}_3) + c_2(\mathrm{e}_1 + \mathrm{e}_2) + c_3(\mathrm{e}_2 + \mathrm{e}_3) + c_4(2\mathrm{e}_4) = 0,$$

we have

$$(c_1 + c_2)\mathrm{e}_1 + (c_2 + c_3)\mathrm{e}_2 + (c_3 + c_1)\mathrm{e}_3 + 2c_4\mathrm{e}_4 = 0.$$

Since $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4\}$ is the standard basis for $\mathbb{R}^4$, it is linearly independent set, we get a homogeneous system of linear equations,

$$\begin{cases} c_1 + c_2 = 0, \\ c_2 + c_3 = 0, \\ c_1 + c_3 = 0, \\ 2c_4 = 0, \end{cases}$$

that is

$$\begin{cases} c_1 = -c_2, \\ c_2 = -c_3, \\ c_3 = -c_1, \\ 2c_4 = 0, \end{cases}$$

equivalently,

$$\begin{cases} c_1 = -c_1, \\ c_2 = c_1, \\ c_3 = -c_1, \\ c_4 = 0. \end{cases}$$

Now $c_1 = -c_1$ if and only if $c_1 = 0$. This implies

$$c_1 = c_2 = c_3 = c_4 = 0.$$

Therefore the set of row vectors $\{e_1+e_3, e_1+e_2, e_2+e_3, 2e_4\}$ is linearly independent and $\det(I+A_\sigma) \neq 0$.

Theorem 3.1.2. Let $\rho \in S_n$ be a cycle of even length. Then $\det(I+A_\rho) = 0$ where A_ρ is a permutation matrix corresponded to ρ.

Proof. If $\rho = (rs)$ is a transposition in S_n where $1 \leq r < s \leq n$, by Lemma 3.1.1, we have that $\det(I+A_\rho) = 0$.

If $\rho = (pqrs)$ where $1 \leq p < q < r < s \leq n$ then we have the permutation matrix corresponded to ρ,

$$A_\rho = \begin{matrix} \\ \\ p\text{- Row} \\ \\ q\text{- Row} \\ \\ r\text{- Row} \\ \\ s\text{- Row} \\ \\ \\ \end{matrix} \begin{bmatrix} e_1 \\ \vdots \\ e_s \\ \vdots \\ e_p \\ \vdots \\ e_q \\ \vdots \\ e_r \\ \vdots \\ e_n \end{bmatrix}, \quad \text{and} \quad I + A_\rho = \begin{matrix} \\ \\ p\text{- Row} \\ \\ q\text{- Row} \\ \\ r\text{- Row} \\ \\ s\text{- Row} \\ \\ \\ \end{matrix} \begin{bmatrix} e_1 \\ \vdots \\ e_p + e_s \\ \vdots \\ e_q + e_p \\ \vdots \\ e_r + e_q \\ \vdots \\ e_s + e_r \\ \vdots \\ e_n \end{bmatrix}.$$

where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$. Note that the rows except, p, q, r, s rows of the matrix $I + A_\rho$, appears 2 in the diagonal positions and 0 in else. To show that the set of vectors

$$\{2e_1, \ldots, (e_p+e_s), \ldots, (e_q+e_p), \ldots, (e_r+e_q), \ldots, (e_s+e_r), \ldots, 2e_n\}$$

is linearly dependent, let $k_1, k_2, \ldots, k_n \in \mathbb{R}$ and

$$2k_1e_1 + 2k_2e_2 + \ldots + k_p(e_p+e_s) + \ldots + k_q(e_q+e_p) + \ldots$$
$$+ k_r(e_r+e_q) + \ldots + k_s(e_s+e_r) + \ldots + 2k_ne_n = 0,$$

thus

$$2k_1\mathrm{e}_1 + 2k_2\mathrm{e}_2 + \ldots + (k_p + k_q)\mathrm{e}_p + \ldots + (k_q + k_r)\mathrm{e}_q + \ldots \\ + (k_r + k_s)\mathrm{e}_r + \ldots + (k_s + k_p)\mathrm{e}_s + \ldots + 2k_n\mathrm{e}_n = 0.$$

Since $\{\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n\}$ is linearly independent, we have a system of homogeneous linear equations

$$\begin{cases} 2k_1 & = 0, \\ \quad 2k_2 & = 0, \\ \qquad \ddots & \vdots \\ \qquad\quad k_p + k_q & = 0, \\ \qquad\qquad \ddots & \vdots \\ \qquad\qquad\quad k_q + k_r & = 0, \\ \qquad\qquad\qquad \ddots & \vdots \\ \qquad\qquad\qquad\quad k_r + k_s & = 0, \\ \qquad\qquad\qquad\qquad \ddots & \vdots \\ \qquad\qquad\qquad\qquad\quad k_s + k_p & = 0, \\ \qquad\qquad\qquad\qquad\qquad \ddots & \vdots \\ \qquad\qquad\qquad\qquad\qquad\quad 2k_n & = 0, \end{cases}$$

that is $k_1 = k_2 = \cdots = k_n = 0$ excepts k_p, k_q, k_r, k_s and

$$\begin{cases} k_p = -k_q, \\ k_q = -k_r, \\ k_r = -k_s, \\ k_s = -k_p. \end{cases}$$

that is

$$\begin{cases} k_p = k_p, \\ k_q = -k_p, \\ k_r = k_p, \\ k_s = -k_p. \end{cases} \tag{3.1.5}$$

Therefore k_p is the parameter of the last system, the system has infinitely many solution. So that the set of vectors

$$\{\mathbf{e}_1, \ldots, (\mathbf{e}_p + \mathbf{e}_s), \ldots, (\mathbf{e}_q + \mathbf{e}_p), \ldots, (\mathbf{e}_r + \mathbf{e}_q), \ldots, (\mathbf{e}_s + \mathbf{e}_r), \ldots, \mathbf{e}_n\}$$

is linearly dependent. By Theorem 2.2.3 assert that $\det(I + A_\rho) = 0$.

Similarly, in any cycle of even length, $\rho := (i_1, i_2, \ldots, i_r)$ where r is even (3.1.5) become

$$\begin{cases} k_{i_1} &= k_{i_1}, \\ k_{i_2} &= -k_{i_1}, \\ k_{i_3} &= k_{i_1}, \\ &\vdots \\ k_{i_r} &= -k_{i_1}. \end{cases}$$

So $\{\mathbf{e}_1, \ldots, (\mathbf{e}_{p_1} + \mathbf{e}_{p_i}), \ldots, (\mathbf{e}_{p_2} + \mathbf{e}_{p_1}), \ldots, (\mathbf{e}_{p_3} + \mathbf{e}_{p_2}), \ldots, (\mathbf{e}_{p_i} + \mathbf{e}_{p_{i-1}}), \ldots, \mathbf{e}_n\}$, is linearly dependent, which implies that $\det(I + A_\rho) = 0$. □

Theorem 3.1.3. Let $\rho \in S_n$ be a cycle of odd length. Then $\det(I + A_\rho) \neq 0$ where A_ρ is permutation matrix corresponded to ρ.

Proof. If $\rho = (rst)$ where $1 \leq r < s < t \leq n$.

We have the permutation matrix corresponded to ρ,

$$A_\rho = \begin{array}{r} \\ \\ r\text{- Row} \\ \\ s\text{- Row} \\ \\ t\text{- Row} \\ \\ \\ \end{array} \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_t \\ \vdots \\ \mathbf{e}_r \\ \vdots \\ \mathbf{e}_s \\ \vdots \\ \mathbf{e}_n \end{bmatrix}, \quad \text{so that} \quad I + A_\rho = \begin{array}{r} \\ \\ r\text{- Row} \\ \\ s\text{- Row} \\ \\ t\text{- Row} \\ \\ \\ \end{array} \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_r + \mathbf{e}_t \\ \vdots \\ \mathbf{e}_s + \mathbf{e}_r \\ \vdots \\ \mathbf{e}_t + \mathbf{e}_s \\ \vdots \\ \mathbf{e}_n \end{bmatrix}.$$

Note that the rows except, r, s, t rows of the matrix $I + A_\rho$, appears 2 in the diagonal positions and 0 in else.

To show that $\{2e_1, 2e_2, \ldots, 2e_{r-1}, (e_r+e_t), \ldots, (e_s+e_r), \ldots, (e_t+e_s), \ldots, 2e_n\}$ is linearly independent.

Let $k_1, k_2, \ldots, k_n \in \mathbb{R}$ and

$$2k_1e_1+2k_2e_2+\ldots+2k_{r-1}e_{r-1}+k_r(e_r+e_t)+\ldots+k_s(e_s+e_r)+\ldots+k_t(e_t+e_s)+2k_ne_n = 0,$$

thus

$$\begin{aligned} 2k_1e_1 + 2k_2e_2 + \ldots + 2k_{r-1}e_{r-1} + (k_r + k_s)e_r + \ldots \\ + (k_s + k_t)e_s + \ldots + (k_r + k_t)e_t + \ldots + 2k_ne_n = 0. \end{aligned}$$

Since $\{e_1, e_2, \ldots, e_n\}$ is linearly independent, we have a system of homogeneous linear equations

$$\begin{cases} 2k_1 & = 0, \\ \quad 2k_2 & = 0, \\ \quad\quad \ddots & \vdots \\ \quad\quad\quad k_r + k_s & = 0, \\ \quad\quad\quad\quad \ddots & \vdots \\ \quad\quad\quad\quad\quad k_s + k_t & = 0, \\ \quad\quad\quad\quad\quad\quad \ddots & \vdots \\ \quad\quad\quad\quad\quad\quad\quad k_r + k_t & = 0, \\ \quad\quad\quad\quad\quad\quad\quad\quad \ddots & \vdots \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad 2k_n & = 0. \end{cases}$$

that is $k_1 = k_2 = \cdots = k_n = 0$ excepts k_r, k_s, k_t and

$$\begin{cases} k_r = -k_s, \\ k_s = -k_t, \\ k_t = -k_r. \end{cases}$$

that is

$$\begin{cases} k_r = -k_r, \\ k_s = k_r, \\ k_t = -k_r. \end{cases} \tag{3.1.6}$$

Since $k_r = -k_r$ implies that $k_r = 0$, and $k_s = 0$. So

$$\{2e_1, 2e_2, \dots, 2e_{r-1}, (e_r + e_t), \dots, (e_s + e_r), \dots, (e_t + e_s), \dots, 2e_n\}$$

is linearly independent. That is $\det(I + A_\rho) \neq 0$.

Similarly, in any cycle of odd length $\rho := (j_1 j_2 \dots j_s)$ where r is odd (3.1.6) become

$$\begin{cases} k_{j_1} = -k_{j_1}, \\ k_{j_2} = k_{j_1}, \\ k_{j_3} = -k_{j_1}, \\ \vdots \\ k_{j_s} = -k_{j_1}, \end{cases}$$

which implies $k_1 = k_1 = \cdots = k_n = 0$, so

$$\{e_1 \dots, (e_{j_1} + e_{j_r}), \dots, (e_{j_2} + e_{j_1}), \dots, (e_{j_3} + e_{j_2}), \dots, (e_{j_s} + e_{j_{s-1}}), \dots, e_n\},$$

is linearly independent, which implies that $\det(I + A_\rho) \neq 0$. □

Theorem 3.1.4. Let $\sigma = \rho_1\rho_2 \dots \rho_t \in S_n$ where $1 \leq t \leq n$ and ρ_ℓ, $1 \leq \ell \leq t$ is one cycle of even length then $\det(I + A_\sigma) = 0$, where A_σ the permutation matrix corresponded to σ.

Proof. Let A_σ be the permutation matrix corresponded to σ. If $\sigma = (pq)(rst)$ where $1 \leq p < q < r < s < t \leq n$, then

$$A_\sigma = \begin{matrix} \\ \\ p\text{-}\,\text{Row} \\ \\ q\text{-}\,\text{Row} \\ \\ r\text{-}\,\text{Row} \\ \\ s\text{-}\,\text{Row} \\ \\ t\text{-}\,\text{Row} \\ \\ \\ \end{matrix} \begin{bmatrix} \mathrm{e}_1 \\ \vdots \\ \mathrm{e}_q \\ \vdots \\ \mathrm{e}_p \\ \vdots \\ \mathrm{e}_t \\ \vdots \\ \mathrm{e}_r \\ \vdots \\ \mathrm{e}_s \\ \vdots \\ \mathrm{e}_n \end{bmatrix}, \quad \text{we have} \quad I + A_\sigma = \begin{matrix} \\ \\ p\text{-}\,\text{Row} \\ \\ q\text{-}\,\text{Row} \\ \\ r\text{-}\,\text{Row} \\ \\ s\text{-}\,\text{Row} \\ \\ t\text{-}\,\text{Row} \\ \\ \\ \end{matrix} \begin{bmatrix} \mathrm{e}_1 \\ \vdots \\ \mathrm{e}_p + \mathrm{e}_q \\ \vdots \\ \mathrm{e}_q + \mathrm{e}_p \\ \vdots \\ \mathrm{e}_r + \mathrm{e}_t \\ \vdots \\ \mathrm{e}_s + \mathrm{e}_r \\ \vdots \\ \mathrm{e}_t + \mathrm{e}_s \\ \vdots \\ \mathrm{e}_n \end{bmatrix},$$

Note that each rows except p, q, r, s, t rows has 2 in diagonal position. To show that

$$\{2\mathrm{e}_1, \dots, (\mathrm{e}_p + \mathrm{e}_q), \dots, (\mathrm{e}_q + \mathrm{e}_p), \dots, (\mathrm{e}_r + \mathrm{e}_t), \dots, (\mathrm{e}_s + \mathrm{e}_r), \dots, (\mathrm{e}_t + \mathrm{e}_s), \dots, 2\mathrm{e}_n\}$$

is linearly dependent. Let $k_1, k_2, \dots, k_n \in \mathbb{R}$ and

$$\begin{aligned} &2k_1\mathrm{e}_1 + 2k_2\mathrm{e}_2 + \dots + k_p(\mathrm{e}_p + \mathrm{e}_q) + \dots + k_q(\mathrm{e}_q + \mathrm{e}_p) + \dots \\ &\qquad + k_r(\mathrm{e}_r + \mathrm{e}_t) + \dots + k_s(\mathrm{e}_s + \mathrm{e}_r) + \dots + k_t(\mathrm{e}_t + \mathrm{e}_s) + \dots 2k_n\mathrm{e}_n = 0, \end{aligned}$$

thus

$$\begin{aligned} &2k_1\mathrm{e}_1 + 2k_2\mathrm{e}_2 + \dots + (k_p + k_q)\mathrm{e}_p + \dots + (k_p + k_q)\mathrm{e}_q + \dots \\ &\qquad + (k_r + k_s)\mathrm{e}_r + \dots + (k_s + k_t)\mathrm{e}_s + \dots + (k_r + k_t)\mathrm{e}_t + \dots + 2k_n\mathrm{e}_n = 0. \end{aligned}$$

Since $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is linearly independent, we have a system of homogeneous linear equations

$$\left\{\begin{array}{lcl} 2k_1 & = & 0, \\ \vdots & & \vdots \\ k_p + k_q & = & 0, \\ \vdots & & \vdots \\ k_p + k_q & = & 0, \\ \vdots & & \vdots \\ k_r + k_s & = & 0, \\ \vdots & & \vdots \\ k_s + k_t & = & 0, \\ \vdots & & \vdots \\ k_r + k_t & = & 0, \\ \vdots & & \vdots \\ 2k_n & = & 0 \end{array}\right.$$

that is $k_1 = k_2 = \cdots = k_n = 0$ excepts k_p, k_q, k_r, k_s, k_t and

$$\left\{\begin{array}{lcl} k_p & = & -k_q, \\ k_q & = & -k_p, \\ k_r & = & -k_s, \\ k_s & = & -k_t, \\ k_t & = & -k_r \end{array}\right.$$

that is

$$\left\{\begin{array}{l} \left\{\begin{array}{lcr} k_p & = & k_p, \\ k_q & = & -k_p, \end{array}\right. \\ \left\{\begin{array}{lcr} k_r & = & -k_r, \\ k_s & = & k_r, \\ k_t & = & -k_r. \end{array}\right. \end{array}\right. \tag{3.1.7}$$

Now consider the two subsystems

$$\left\{\begin{array}{lcr} k_p & = & k_p, \\ k_q & = & -k_p, \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcr} k_r & = & -k_r, \\ k_s & = & k_r, \\ k_t & = & -k_r. \end{array}\right.$$

The subsystem

$$\begin{cases} k_r &= -k_r, \\ k_s &= k_r, \\ k_t &= -k_r. \end{cases}$$

has unique solution $k_r = k_s = k_t = 0$, since $k_r = -k_r$ if and only if $k_r = 0$. But the subsystem

$$\begin{cases} k_p &= k_p, \\ k_q &= -k_p, \end{cases}$$

has infinitely many solution. Therefore $\{e_1, \ldots, (e_p + e_q), \ldots, (e_q + e_p), \ldots, (e_r + e_t), \ldots, (e_s+e_r), \ldots, (e_t+e_s), \ldots, e_n\}$ is linearly dependent. That is $\det(I+A_\sigma) = 0$.

In general, we consider (3.1.7) in t-subsystem of homogeneous linear equations correspond to $\rho_1, \rho_2, \ldots, \rho_t$, we see that if there exists some subsystem corresponding to ρ_j say, which is of even length then the full system must has infinitely many solution. Therefore $\det(I + A_\sigma) = 0$. If there is no such any cycles of even length in the permutation σ then $\det(I + A_\sigma) \neq 0$. □

Theorem 3.1.5. If A and B are permutation matrices in $P(n)$ and $A^{-1}B$ corresponding to permutation $\sigma = \rho_1\rho_2 \ldots \rho_k \in S_n$ then

a) $A + B$ is singular, when $\exists \rho_i, 1 \leq i \leq k$ has even length.

b) $A + B$ is nonsingular, when $\forall \rho_j$, $1 \leq i \leq k$ has odd length.

Proof. Since A is a permutation matrix, it is an orthogonal matrix, then A^{-1} exists. Consider

$$A^{-1}(A + B) = A^{-1}A + A^{-1}B = I + A^{-1}B := I + C$$

where $C = A^{-1}B \in P(n)$.

If C is permutation matrix corresponding a permutation $\sigma = \rho_1\rho_2 \ldots \rho_k \in S_n$ and $\exists \rho_i, 1 \leq i \leq k$ has even length, Theorem 3.1.4 assert that $\det(I + C) = 0$.

Since the matrix $A + B$ is equivalent to the matrix $I + C$ therefore

$$\operatorname{rank}(A + B) = \operatorname{rank}(I + C),$$

by Theorem 2.2.4. Thus $\det(A + B) = 0$, this prove a).

Similarly, if $\forall \rho_j$, $1 \leq i \leq k$ has odd length then Theorem 3.1.4 also assert that $\det(I + C) \neq 0$ that is $\det(A + B) \neq 0$, the case b) was proved. □

3.2 Linear Combination of Permutation Matrices

Theorem 3.2.1. If A and B are permutation matrices in $P(n)$ and $A^{-1}B$ corresponding to permutation $\sigma = \rho_1\rho_2 \ldots \rho_k \in S_n$, and $c_1, c_2 \in \mathbb{R} \setminus \{0\}$, $c_1 + c_2 \neq 0$, then

a) $c_1A + c_2B$ is singular, when $\exists \rho_i, 1 \leq i \leq k$ has even length.

b) $c_1A + c_2B$ is nonsingular, when $\forall \rho_j$, $1 \leq i \leq k$ has odd length.

Proof. Consider the combination $c_1A + c_2B$ of the permutation $A, B \in P(n)$. In a special case, if $A = B$ and $c_1 + c_2 = 0$ then we have $c_1A + c_2B = c_1A - c_1A = 0$. Therefore the combination matrix $c_1A + c_2B$ is zero matrix.

In general, if $c_1, c_2 \in \mathbb{R} \setminus \{0\}$, $c_1 + c_2 \neq 0$ consider the matrix $c_1A + c_2B$. Since A is permutation matrix, then $(c_1A)^{-1}$ exists and $(c_1A)^{-1} = (1/c_1)A^{-1}$. Consider

$$(c_1A)^{-1}(c_1A + c_2B) = (c_1/c_1)A^{-1}A + (c_2/c_1)A^{-1}B = I + (c_2/c_1)A^{-1}B := I + D$$

where $D = (c_2/c_1)A^{-1}B \in P(n)$.

If D is permutation matrix corresponding a permutation $\sigma = \rho_1\rho_2 \ldots \rho_k \in S_n$ and $\exists \rho_i, 1 \leq i \leq k$ has even length, Theorem 3.1.4 assert that $\det(I + D) = 0$.

Since the matrix $A + B$ is equivalent to the matrix $I + D$ therefore

$$\text{rank}(A + B) = \text{rank}(I + D),$$

by Theorem??. Thus $\det(A + B) = 0$, this prove a).

Similarly, if $\forall \rho_j$, $1 \leq i \leq k$ has odd length then Theorem 3.1.4 also assert that $\det(I + D) \neq 0$ that is $\det(A + B) \neq 0$, the case b) was proved. □

Lemma 3.2.2. Let B is a reflection matrix corresponded to $\sigma = \rho_1\rho_2\ldots\rho_k$ for some $k \in \mathbb{N}$. Then there is at lest one ρ_j of even length where $1 \leq j \leq k$.

Proof. Suppose that B is reflection matrix corresponded to $\sigma = \rho_1\rho_2\ldots\rho_j$. Then $\det(B) = -1$. Assume that there is no any even length of σ, that is σ consists of all odd length. Thus it is even permutation, so $\det(B) = 1$, which is a contradiction. Hence there is at lest one ρ_j for some $1 \leq j \leq k$, of even length. □

Corollary 3.2.3. If A is rotation matrix and B is reflection matrix, then

$$\det(A + B) = \det A + \det B.$$

Proof. By Theorem 3.1.4, $\det(A + B) = 0$. Since $\det(A) = 1$ and $\det(B) = -1$, we have

$$\det(A + B) = 0 = -1 + 1 = \det(A) + \det(B).$$

□